Exam II solutions
Topology (Math 5853)

Choose four problems. If you do five, please say which four you want graded.

1(a) Show that every closed subspace of a compact space is compact.
(b) Show that every compact Hausdorff space is regular.

SOLUTION.

(A) Let $Y \subset X$ be a closed subspace. Let $\mathcal{A}$ be a covering of Y by open sets of X. Note that $X - Y$ is open, so $\mathcal{A}' = \mathcal{A} \cup \{X - Y\}$ is an open covering of X. By compactness of X, there is a finite subcovering $\{A_1, \ldots, A_n\} \subset \mathcal{A}'$. Now, $\{A_1, \ldots, A_n\} \cap \mathcal{A}$ is a finite subset of $\mathcal{A}$ which covers Y (it equals $\{A_1, \ldots, A_n\}$ if $X - Y$ is not in this set, otherwise it is this collection with $X - Y$ removed). Hence Y is compact.

(B) Let X be a compact Hausdorff space. First, one-point sets in X are closed because X is Hausdorff. Next, let x and B be given, where $x \in X$ and B is a closed set not containing x. For each $b \in B$ there exist disjoint open sets U_b, V_b such that $x \in U_b$ and $b \in V_b$. The sets $\{V_b\}$ are an open covering of B. By part (a) B is compact, so there is a finite subcovering $\{V_{b_1}, \ldots, V_{b_n}\}$. Now let $V = V_{b_1} \cup \cdots \cup V_{b_n}$ and $U = U_{b_1} \cap \cdots \cap U_{b_n}$. These sets are open and disjoint, and $x \in U$ and $B \subset V$. Hence X is regular.

2. Let $p\colon X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected and Y is connected, then X is connected.

SOLUTION. Suppose U, V are disjoint open sets in X whose union is X. For each $y \in Y$ the set $p^{-1}(y)$ is connected, so it is entirely contained in one of U and V. Hence, U and V are both saturated sets (equal to unions of preimage sets $p^{-1}(y)$). Since p is a quotient map, $p(U)$ and $p(V)$ are open in Y. Since p is surjective, their union is Y. They are disjoint because their preimages are U and V which are disjoint (this uses the fact that U and V are saturated). Hence, since Y is connected, one of $p(U)$ and $p(V)$ must be empty. This implies that one of U and V is empty. Hence X is connected.

3(a) Suppose X is locally compact and Hausdorff, but not compact. Define the topology for the one-point compactification $X \cup \{\infty\}$. (That is, name all the open sets.)
(b) Give an example of two non-homeomorphic locally compact Hausdorff spaces whose one-point compactifications are homeomorphic.

SOLUTION.

(A) There are two types of open sets in $X \cup \{\infty\}$. Type I sets are the open subsets of X. Type II sets are all sets of the form $(X - C) \cup \{\infty\}$, where C is a compact subset of X.

(B) If you start with a compact Hausdorff space Z and delete a point, the resulting space is locally compact and Hausdorff, and has one-point compactification homeomorphic to Z.

There are many possible examples. Let Z be a figure-eight in the plane, for example the union of the two circles of radius one, centered at $(\pm 1, 0)$.

Let $X = Z - \{(0,0)\}$ and let $Y = Z - \{(2,0)\}$. Then X is not connected (it has a separation given by the left and right open half-planes in $\mathbb{R}^2$) and Y is connected. Hence, X and Y are not homeomorphic. To see that Y is connected, it is not hard to show that it is path connected (which implies connected). Every point can easily be joined to the origin by a path in Y.

UPDATE: here is a simpler example: take $Z = [0,2]$ and $X = [0,1) \cup (1,2]$, $Y = [0,2)$. Clearly Y is connected and X is not.

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4. Give the details of the argument showing that if X and Y are connected then so is $X \times Y$. State carefully any results that you use.

SOLUTION. First, the sets $X \times \{y\}$ and $\{x\} \times Y$ are connected, being homeomorphic to X and Y respectively. Pick a basepoint $(x_0, y_0) \in X \times Y$. For any $y \in Y$, the set $T_y = (X \times \{y\}) \cup (\{x_0\} \times Y)$ is connected, since it is a union of two connected subspaces of $X \times Y$ which have a point in common. Note that the basepoint is contained in every subspace T_y. Finally, the union $\bigcup_{y \in Y} T_y$ is connected, since each T_y is connected and they share a common point. This union is all of $X \times Y$, and so $X \times Y$ is connected.

Results used: the union of a collection of connected sets that have a point in common is connected (Theorem 23.3).

5(a) Give the definitions of *components* and *path components* of a space X.
(b) What is the precise relationship between components and path components of a space? Give proofs and/or counterexamples for your assertions.

SOLUTION.

(A) Components of X are equivalence classes of the relation $\sim$, where $x \sim y$ if there is a connected subspace containing x and y. Path components are equivalence classes for the relation $\sim_p$, where $x \sim_p y$ if there is a path in X from x to y (that is, if there is a continuous map $f\colon [0,1] \to X$ with $f(0) = x$ and $f(1) = y$).

(B) Every path component is contained in a component, but not conversely. Thus, every component is a disjoint union of path components.

For the first assertion, note that if $x \sim_p y$ then let f be a path in X from x to y. Since $[0,1]$ is connected and f is continuous, the image $f([0,1])$ is connected, and so $f(0) \sim f(1)$. That is, $x \sim_p y$ implies $x \sim y$.

For the second claim, the topologist's sine curve is an example of a space which is connected, but has two path components.
