Exam I Solutions
Topology (Math 5853)

1(a) State the axioms for $\mathscr{B}$ to be a basis.
(b) Define the topology $\mathscr{T}$ generated by $\mathscr{B}$.
(c) Suppose $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases generating the topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively on a set X. State a necessary and sufficient criterion in terms of $\mathscr{B}_1$ and $\mathscr{B}_2$ for $\mathscr{T}_1$ to be finer than $\mathscr{T}_2$.

Solution.

(a) A basis $\mathscr{B}$ is a collection of subsets of X such that (i) their union is all of X, and (ii) given $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$ there is a $B \in \mathscr{B}$ such that $x \in B$ and $B \subset B_1 \cap B_2$.

(b) The topology $\mathscr{T}$ defined by $\mathscr{B}$ is defined as follows. A subset U is in $\mathscr{T}$ (ie. is open) if, for every $x \in U$ there is a $B \in \mathscr{B}$ such that $x \in B$ and $B \subset U$. Equivalently, $U \in \mathscr{T}$ if and only if U is a union of elements of $\mathscr{B}$.

(c) $\mathscr{T}_1$ is finer than $\mathscr{T}_2$ if and only if $\mathscr{T}_1 \supset \mathscr{T}_2$. In terms of $\mathscr{B}_1$ and $\mathscr{B}_2$, this means that for every $B_2 \in \mathscr{B}_2$ and $x \in B_2$ there is a $B_1 \in \mathscr{B}_1$ such that $x \in B_1$ and $B_1 \subset B_2$.

2(a) Say what it means for an ordered set A to be *well-ordered*.
(b) Define the *dictionary ordering* on $A \times B$, where A and B are ordered sets.
(c) Show that if A and B are well-ordered, then so is $A \times B$ (in the dictionary order).

Solution.

(a) An ordered set A is well-ordered if every non-empty subset of A has a smallest element.

(b) The dictionary ordering on $A \times B$ is defined by: $(a, b) < (a', b')$ if $a < a'$ or $a = a'$ and $b < b'$.

(c): Let $S \subset A \times B$ be a non-empty subset. Define $A_0 \subset A$ to be the set of first coordinates of elements of S: $A_0 = \{a \in A \mid (a, b) \in S \text{ for some } b \in B\}$. Note that A_0 is non-empty, hence it has a smallest element a_0. Next let $B_0 = \{b \in B \mid (a_0, b) \in S\}$. Note that B_0 is a non-empty subset of B. Let b_0 be the smallest element of B_0. Then, (a_0, b_0) is the smallest element of S. (If $(a, b) \in S$ then either $a_0 < a$ and hence $(a_0, b_0) < (a, b)$, or $a_0 = a$. But then $b \in B_0$, and so $b_0 \leq b$ and therefore $(a_0, b_0) \leq (a, b)$.)

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3. Let $f \colon \mathbb{R} \to \mathbb{R}^\omega$ be given by $f(t) = (t, \frac{1}{2}t, \frac{1}{4}t, \frac{1}{8}t, \ldots)$. Show that f is not continuous if $\mathbb{R}^\omega$ is given the box topology.

Solution. The coordinate functions of f are given by $f_n(t) = \frac{1}{2^{n-1}}t$. Note that $f_n^{-1}(t) = 2^{n-1}t$. So, the pre-image of the interval $(-a, a)$ under f_n is $(-2^{n-1}a, 2^{n-1}a)$.

Let U be the open set $(-1, 1) \times (-\frac{1}{2 \cdot 2}, \frac{1}{2 \cdot 2}) \times (-\frac{1}{3 \cdot 2^2}, \frac{1}{3 \cdot 2^2}) \times (-\frac{1}{4 \cdot 2^3}, \frac{1}{4 \cdot 2^3}) \times \cdots$. The nth interval I_n is $(-\frac{1}{n \cdot 2^{n-1}}, \frac{1}{n \cdot 2^{n-1}})$, which has pre-image $(-\frac{1}{n}, \frac{1}{n})$ under f_n.

Now, $f^{-1}(U) = f^{-1}(I_1 \times I_2 \times \cdots) = \{t \in \mathbb{R} \mid f_n(t) \in I_n \text{ for all } n\}$, which is the intersection of the sets $f_n^{-1}(I_n)$. So $f^{-1}(U) = (-1,1) \cap (-\frac{1}{2}, \frac{1}{2}) \cap (-\frac{1}{3}, \frac{1}{3}) \cap \cdots = \{0\}$. Since this set is not open in $\mathbb{R}$, the function f is not continuous.

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4. Let Y be a Hausdorff space. Suppose $g, h\colon X \to Y$ are continuous maps. Prove that the set $\{x \in X \mid g(x) = h(x)\}$ is closed. [Hint: use $Y \times Y$.]

Solution 1. Define $f\colon X \to Y \times Y$ by $f(x) = g(x) \times h(x)$. Since its component functions g and h are continuous, so is f. Recall that a space Y is Hausdorff if and only if the diagonal Δ is a closed subset of $Y \times Y$. Hence, Δ is closed, and by continuity, $f^{-1}(\Delta)$ is closed. Finally, note that $f(x) \in \Delta$ if and only if $g(x) = h(x)$. Therefore the closed set $f^{-1}(\Delta)$ is exactly $\{x \in X \mid g(x) = h(x)\}$.

Solution 2. We will show that the set $S = \{x \in X \mid g(x) \neq h(x)\}$ is open in X. If $x \in S$ then $g(x)$ and $h(x)$ are distinct points in Y. Since Y is Hausdorff, there exist disjoint open sets U, V in Y such that $g(x) \in U$ and $h(x) \in V$. Since g and h are continuous, the sets $g^{-1}(U)$ and $h^{-1}(V)$ are both open neighborhoods of x. Let $W = g^{-1}(U) \cap h^{-1}(V)$, an open neighborhood of x. We claim that $W \subset S$, which implies that S is open. To see this, if $w \in W$ then $g(w) \in U$ and $h(w) \in V$, and therefore $g(w) \neq h(w)$ since these sets are disjoint in Y. Hence $w \in S$.

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5. Let $X = \mathbb{Z} \times [0,1]$ and define an equivalence relation $\sim$ on X by: $(n,1) \sim (n+1,0)$ for all $n \in \mathbb{Z}$ (no other identifications are made).
(a) Draw a picture of X and also indicate what the quotient space X^* looks like.
(b) Show that there is a continuous bijection $X^* \to \mathbb{R}$. State carefully what needs to be verified in order to define this function and know that it is continuous. Verify that the required properties hold.

Solution.
(a) The set X is a union of disjoint copies of the interval, like this:

$$\cdots \quad | \quad | \quad | \quad | \quad | \quad | \quad | \quad | \quad \cdots$$

The equivalence relation identifies the top endpoint of each interval with the lower endpoint of the next interval. Thus, the intervals get joined end to end, to form a line.

(b) Define a function $g\colon \mathbb{Z} \times [0,1] \to \mathbb{R}$ by $g(n,t) = n + t$. Note that $g(n,1) = g(n+1,0)$ and so g is constant on equivalence classes. Hence, g induces a well defined function $f\colon X^* \to \mathbb{R}$. According to a theorem from class (Theorem 22.2) f is continuous if and only g is continuous. The latter function is continuous because it is the restriction of a continuous function $\mathbb{R}^2 \to \mathbb{R}$.

For any $r \in \mathbb{R}$ we can write $r = n + t$ where n is the largest integer less than or equal to r, and $t \in [0,1)$. Hence, f is surjective. f is injective as follows: if $n + t = n' + t'$ with $n, n' \in \mathbb{Z}$ and $t, t' \in [0,1]$ then either $n = n'$ (which implies $t = t'$) or $n < n'$ or $n' < n$. If $n < n'$ then t and t' differ by at least 1, since n, n' are integers. Since $t, t' \in [0,1]$ it must be the case that $t' = 0$ and $t = 1$. But then $(n,t) \sim (n',t')$. The case $n' < n$ is similar. We have shown that $g(n,t) = g(n',t')$ implies $(n,t) \sim (n',t')$, which means that the induced function f is injective.

Alternatively, one can use Corollary 22.3 and observe that the equivalence classes for $\sim$ are exactly the pre-images under g of points of $\mathbb{R}$. Then, the corollary says that f is a bijection.
