Final Exam Solutions
Topology I (Math 5853)

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1. Let $\mathbb{R}'$ be the set $\mathbb{R}$ with the topology given by the basis $\mathcal{B} = \{[a,b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}$. Determine the closures of the following sets in $\mathbb{R}'$:
(a) $A = (0, \sqrt{2})$
(b) $B = (\sqrt{2}, 3)$

SOLUTION.
(A) $\overline{A} = [0, \sqrt{2}]$. We claim that 0 and $\sqrt{2}$ are limit points of A, and that no other points outside of A are limit points. Every basic open neighborhood of 0 has the form $[a, b)$ with a, b rational and $a \le 0 < b$. Then $b/2$ is in the intersection $[a, b) \cap A$, so 0 is a limit point. Every basic open neighborhood of $\sqrt{2}$ has the form $[a, b)$ with a, b rational and $a < \sqrt{2} < b$, because $\sqrt{2}$ is irrational. Now any number x between a and $\sqrt{2}$ is in the intersection $[a, b) \cap A$, and so $\sqrt{2}$ is a limit point.

If $x > \sqrt{2}$ then there is a rational r between x and $\sqrt{2}$, so $[r, x + 1)$ is a neighborhood of x disjoint from A. If $x < 0$ then $[r, 0)$ is a neighborhood of x disjoint from A, for any rational $r < x$.

(B) $\overline{B} = [\sqrt{2}, 3)$. We claim that $\sqrt{2}$ is the only limit point of B outside of B. Every basic open neighborhood of $\sqrt{2}$ has the form $[a, b)$ with a, b rational and $a < \sqrt{2} < b$. Let x be a rational between $\sqrt{2}$ and b, and less than 3. Then x is in $[a, b) \cap B$, so $\sqrt{2}$ is a limit point.

If $x \ge 3$ then $[x, x + 1)$ is a neighborhood of x disjoint from B. If $x < \sqrt{2}$ then there exist rational numbers r, s such that $r < x < s < \sqrt{2}$. Then $[r, s)$ is a neighborhood of x that is disjoint from B.

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2. Prove the *Tube Lemma*: Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the subset $x_0 \times Y$, then x_0 has a neighborhood W in X such that $W \times Y$ is contained in N.

SOLUTION. This is Lemma 26.8 of Munkres.

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3. Prove the following lemma: if $f\colon X \to Y$ is a continuous injective map and X is compact and Y is Hausdorff, then f is an embedding.

SOLUTION. Let $g\colon X \to f(X)$ be the same as f, but with restricted range. Now g is a continuous bijection. Note that $f(X)$ is Hausdorff since it is a subspace of the Hausdorff space Y. We need to show that g^{-1} is continuous for f to be an embedding.

Let $C \subset X$ be a closed set. Since X is compact, so is C. Since g is continuous, $g(C)$ is compact. Since $f(X)$ is Hausdorff, $g(C)$ is closed in $f(X)$. This shows g^{-1} is continuous.

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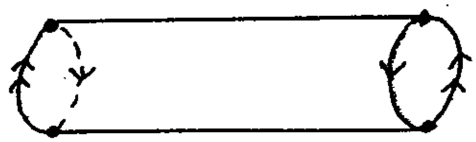
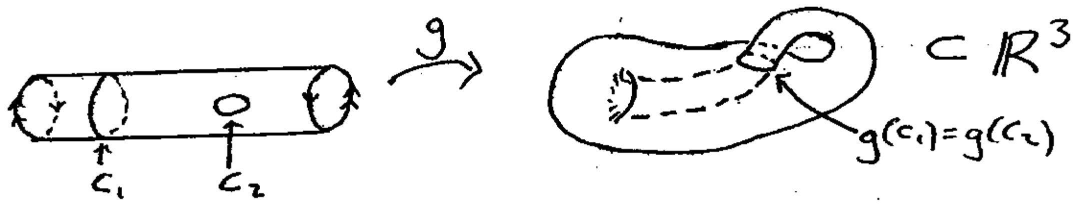
4. Let $r\colon S^1 \to S^1$ be a reflection of the circle (e.g. $(x, y) \mapsto (-x, y)$ in the plane). The *Klein bottle* K is the quotient space of $[0, 1] \times S^1$ under the following equivalence relation: $(0, z) \sim (1, r(z))$ for all $z \in S^1$, and (t, z) is not equivalent to anything except itself, for $t \neq 0, 1$. [That is, glue one boundary circle to the other, using the reflection r to join them. The reflecion means that you won't get a torus.]



(a) Explain why K is compact.

(b) Let $C_1 \subset K$ be (the image of) the circle $\{\frac{1}{3}\} \times S^1$, and let $C_2 \subset K$ be a small embedded circle inside $(\frac{1}{2}, \frac{3}{4}) \times S^1$ as in the picture. There is a continuous map $g\colon K \to \mathbb{R}^3$ as shown in the picture, which is *almost* injective. Specifically, the restriction of g to $K - C_1$ is injective, and so is the restriction to $K - C_2$.



Assuming g exists as described, use Urysohn's Lemma to construct a continuous map of K into $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ which is an embedding. You may assume that K is Hausdorff.

SOLUTION.

(A) We know the spaces $[0, 1]$ and S^1 are compact, so their product $[0, 1] \times S^1$ is also compact. From the definition of K, there is a quotient map $p\colon [0, 1] \times S^1 \to K$. Quotient maps are continuous and surjective, so K is the image under p of the compact space $[0, 1] \times S^1$. Hence K is compact. (So: every quotient space of a compact space is compact.)

(B) Since K is Hausdorff and C_1 and C_2 are homeomorphic to S^1 and therefore are compact, we can conclude that C_1 and C_2 are closed sets in K. They are also disjoint. Since K is compact and Hausdorff, it is normal, and so the Urysohn Lemma applies. It says there is a continuous function $f\colon K \to [0, 1]$ which takes C_1 to $\{0\}$ and C_2 to $\{1\}$. Composing with the inclusion map $[0, 1] \to \mathbb{R}$, there is a continuous function $f'\colon K \to \mathbb{R}$ with $f'(C_1) = \{0\}$ and $f'(C_2) = \{1\}$.

Define $h\colon K \to \mathbb{R}^3 \times \mathbb{R}$ by $h(x) = (g(x), f'(x))$. This is continuous, since its coordinate functions g and f' are continuous. Also, h is injective. Consider two distinct points $x, y \in K$. If x and y are in C_1 and C_2, then $f'(x) \neq f'(y)$ and so $h(x) \neq h(y)$. Otherwise, at least one of C_1, C_2 does not contain x or y. Without loss of generality suppose C_1 does not contain x or y. Then $x, y \in K - C_1$, and g is injective on this subset. So, $g(x) \neq g(y)$ and hence $h(x) \neq h(y)$.

Finally, since h is a continuous injective map from a compact space to a Hausdorff space, it is an embedding by problem 3.

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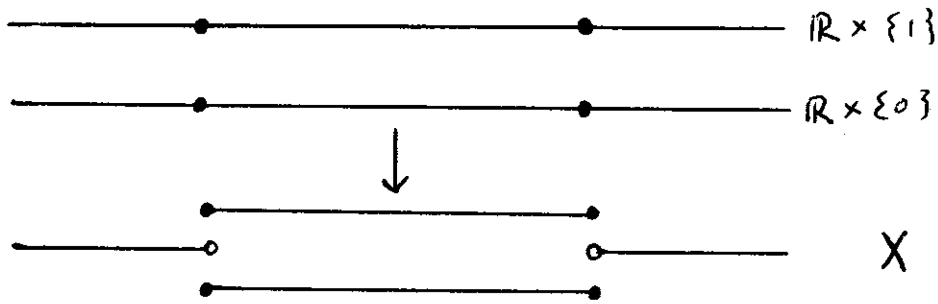
5. Let X be the quotient space obtained from $\mathbb{R} \times \{0, 1\}$ by identifying $x \times 0$ with $x \times 1$ for every number x with $|x| > 1$. [You may want to draw a picture. Think about which sets in X are open sets in the quotient topology.]

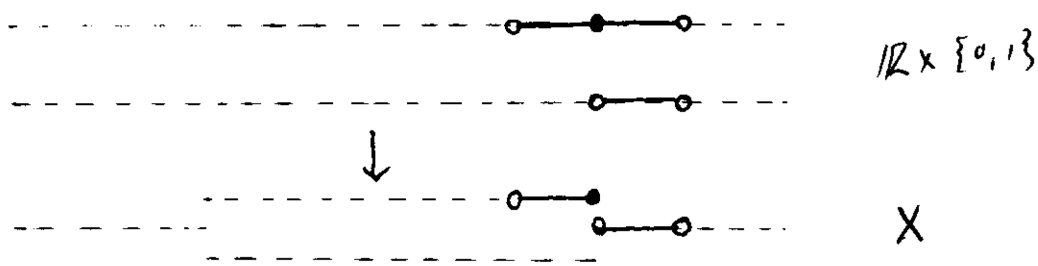
(a) Does X satisfy the T_1 axiom? Why or why not?

(b) Is X Hausdorff? Why or why not?

SOLUTION. This problem is quite tricky. The key is to understand the open neighborhoods of the points of X correspoonding to $\pm 1 \times 0$ and $\pm 1 \times 1$. Remember, open sets in X are the images of *saturated* open sets in $\mathbb{R} \times \{0, 1\}$. Here is a picture of the underlying set for X:



Here is a small open neighborhood of $[1 \times 1]$ (explanation in (b) below):



All points other than the four "corners" have neighborhoods homeomorphic to ordinary intervals.

(A) To show that X is a T_1 space, let x and y be distinct points in X. If their $\mathbb{R}$-coordinates differ (note, X is not a product, but the $\mathbb{R}$-coordinate still makes sense) then they can in fact be separated by open sets (as in the Hausdorff property). If x and y are of the form $\{[r \times 0], [r \times 1]\}$ with $|r| < 1$ then they can be separated by disjoint open neighborhoods which are the images of the sets $U \times \{0\}$ and $U \times \{1\}$ for some $U \subset (-1, 1)$.

If $\{x, y\} = \{[1 \times 0], [1 \times 1]\}$ then each of x, y has a neighborhood, as in the picture above, not containing the other. Similarly for $\{x, y\} = \{[-1 \times 0], [-1 \times 1]\}$.

(B) X is not Hausdorff. Use the points $[1 \times 0]$ and $[1 \times 1]$. Every neighborhood of $[1 \times 1]$ must contain a set as shown in the picture. Its preimage in $\mathbb{R} \times \{0, 1\}$ contains an interval $(1-\epsilon, 1+\epsilon) \times \{1\}$ around 1×1, and since it is saturated, it must also contain $(1, 1+\epsilon) \times \{0\}$. Neighborhoods of the point $[1 \times 0]$ must contain a similar set. Two such neighborhoods will always have points in common.

6. Let X be a compact metric space and suppose that $f\colon X \to X$ is an *isometry*: $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that f is a homeomorphism. [Hint for surjectivity: if not, construct a sequence having no limit point.]

SOLUTION. For injectivity, if $f(x) = f(y)$ then $d(f(x), f(y)) = 0$. Hence $d(x, y) = 0$ which implies that $x = y$.

For continuity, note that given $x \in X$ and $\epsilon > 0$, let $\delta = \epsilon$. Then, $d(x,y) < \delta$ implies $d(f(x), f(y)) < \epsilon$ because $d(x,y) = d(f(x), f(y))$. Hence f is continuous (by the metric space characterization of continuity).

Note that X is Hausdorff, being a metric space. So compact sets are the same as closed sets. Since f is continuous, $f(C)$ is compact, and closed. If f is not surjective, then $X - f(X)$ is a non-empty open set. Pick a point $z \in X - f(X)$ and an open ball $B(z, \delta)$ disjoint from $f(X)$.

Define $z_1 = f(z)$, $z_2 = f(z_1)$, $z_3 = f(z_2)$, and so on. For any $m < n$ we have

$$d(z_m, z_n) = d(z_{m-1}, z_{n-1}) = \cdots = d(z, z_{n-m})$$

and the latter distance is at least δ because $z_{n-m} \in f(X)$. Thus, every pair of points in the set $\{z_i \mid i \in \mathbb{Z}_+\}$ has distance δ or greater. This is an infinite subset of a compact space which has the discrete topology, which is a contradiction. Hence, f is surjective.

Finally, since f is a bijection, it is a homeomorphism by problem 3. (Alternatively, f^{-1} is continuous for the same reason as f.)

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7. Let $A \subset \mathbb{R}^\omega$ be defined by

$$A = \{(x_i) \in \mathbb{R}^\omega \mid x_i = 0 \text{ for all but finitely many } i\}.$$

(a) Prove that A is dense in $\mathbb{R}^\omega$ with the product topology.

(b) Prove that A is not dense in $\mathbb{R}^\omega$ with the box topology.

Solution.

(a) Let (x_i) be a point in $\mathbb{R}^\omega$ and let $U = \prod_i U_i$ be a basic open neighborhood of (x_i). Then $U_i = \mathbb{R}$ for all but finitely many i. Define the point (y_i) by letting $y_i = 0$ if $U_i = \mathbb{R}$ and $y_i = x_i$ otherwise. Then (y_i) is in U and in A. Hence (x_i) is a limit point of A. Since (x_i) was arbitrary, A is dense.

(b) In the box topology the set

$$U = (1,2) \times (1,2) \times (1,2) \times \cdots$$

is an open set, and it is non-empty. It is also disjoint from A, since no point of U has any zero coordinates. Hence, A is not dense; every point of U fails to be a limit point of A.
