Final Exam
Topology I (Math 5853)
December 11, 2013

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**1.** Let $\mathbb{R}'$ be the set $\mathbb{R}$ with the topology given by the basis $\mathcal{B} = \{[a, b) \mid a < b \text{ and } a, b \in \mathbb{Q}\}$. Determine the closures of the following sets in $\mathbb{R}'$:

**(a)** $A = (0, \sqrt{2})$

**(b)** $B = (\sqrt{2}, 3)$

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**2.** Prove the *Tube Lemma*: Consider the product space $X \times Y$ where $Y$ is compact. If $N$ is an open set of $X \times Y$ containing the subset $x_0 \times Y$, then $x_0$ has a neighborhood $W$ in $X$ such that $W \times Y$ is contained in $N$.

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**3.** Prove the following lemma: if $f\colon X \to Y$ is a continuous injective map and $X$ is compact and $Y$ is Hausdorff, then $f$ is an embedding.

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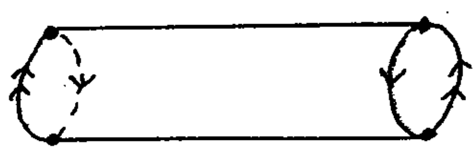
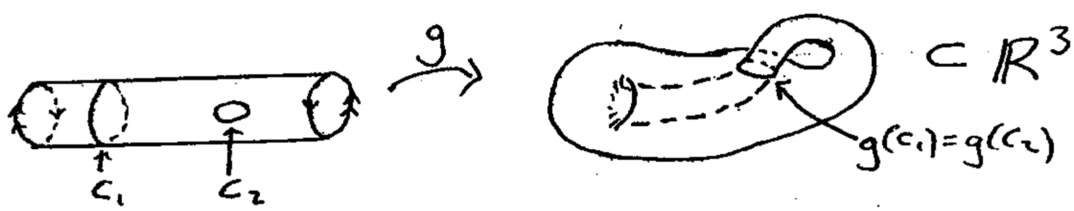
**4.** Let $r\colon S^1 \to S^1$ be a reflection of the circle (e.g. $(x, y) \mapsto (-x, y)$ in the plane). The *Klein bottle* $K$ is the quotient space of $[0, 1] \times S^1$ under the following equivalence relation: $(0, z) \sim (1, r(z))$ for all $z \in S^1$, and $(t, z)$ is not equivalent to anything except itself, for $t \neq 0, 1$. [That is, glue one boundary circle to the other, using the reflection $r$ to join them. The reflecion means that you won't get a torus.]



**(a)** Explain why $K$ is compact.

**(b)** Let $C_1 \subset K$ be (the image of) the circle $\{\frac{1}{3}\} \times S^1$, and let $C_2 \subset K$ be a small embedded circle inside $(\frac{1}{2}, \frac{3}{4}) \times S^1$ as in the picture. There is a continuous map $g\colon K \to \mathbb{R}^3$ as shown in the picture, which is *almost* injective. Specifically, the restriction of $g$ to $K - C_1$ is injective, and so is the restriction to $K - C_2$.




Assuming $g$ exists as described, use Urysohn's Lemma to construct a continuous map of $K$ into $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ which is an embedding. You may assume that $K$ is Hausdorff.

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**5.** Let $X$ be the quotient space obtained from $\mathbb{R} \times \{0,1\}$ by identifying $x \times 0$ with $x \times 1$ for every number $x$ with $|x| > 1$. [You may want to draw a picture. Think about which sets in $X$ are open sets in the quotient topology.]

**(a)** Does $X$ satisfy the $T_1$ axiom? Why or why not?

**(b)** Is $X$ Hausdorff? Why or why not?

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**6.** Let $X$ be a compact metric space and suppose that $f\colon X \to X$ is an *isometry*: $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that $f$ is a homeomorphism. [Hint for surjectivity: if not, construct a sequence having no limit point.]

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**7.** Let $A \subset \mathbb{R}^\omega$ be defined by

$$A = \{(x_i) \in \mathbb{R}^\omega \mid x_i = 0 \text{ for all but finitely many } i\}.$$

**(a)** Prove that $A$ is dense in $\mathbb{R}^\omega$ with the product topology.

**(b)** Prove that $A$ is not dense in $\mathbb{R}^\omega$ with the box topology.

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