Final Exam
Topology (Math 5853)
December 15, 2005

1. **(a)** Show that every closed subspace of a compact space is compact.

(b) Show that every compact Hausdorff space is regular.

(c) Show that every compact Hausdorff space is normal.

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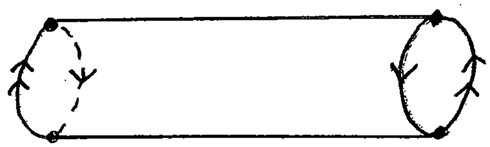
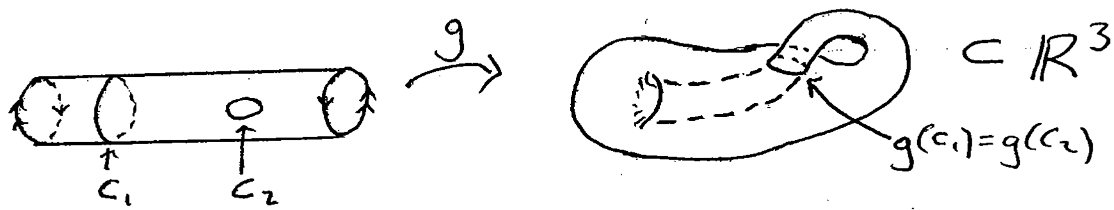
2. Let $r\colon S^1 \to S^1$ be a reflection of the circle (e.g. $(x, y) \mapsto (-x, y)$ in the plane). The *Klein bottle* K is the quotient space of $[0, 1] \times S^1$ under the following equivalence relation: $(0, z) \sim (1, r(z))$ for all $z \in S^1$, and (t, z) is not equivalent to anything except itself, for $t \neq 0, 1$.



(a) Explain why K is compact.

(b) Prove the following general fact: if $f\colon X \to Y$ is a continuous injective map and X is compact and Y is Hausdorff, then f is an embedding.

(c) Let $C_1 \subset K$ be (the image of) the circle $\{\frac{1}{3}\} \times S^1$, and let $C_2 \subset K$ be a small embedded circle inside $(\frac{1}{2}, \frac{3}{4}) \times S^1$. As shown in the picture below, there is a continuous map $g\colon K \to \mathbb{R}^3$ which is almost injective: the restriction of g to $K - C_1$ is injective, and so is the restriction of g to $K - C_2$.



Assuming g exists, construct an embedding of K into $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. State carefully any theorems that you use. You may also assume that K is Hausdorff.

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3. Let X be Hausdorff and locally compact, but not compact.

(a) What are the open sets in $X \cup \{\infty\}$, the one-point compactification of X? Prove that these sets form a topology.

(b) Prove that $X \cup \{\infty\}$ is compact.

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4. Prove that if X and Y are connected then so is $X \times Y$. State any lemmas that you need, and prove them afterwards if you can.

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5. Let $A \subset \mathbb{R}^\omega$ be defined by

$$A = \{(x_i) \in \mathbb{R}^\omega \mid x_i = 0 \text{ for all but finitely many } i\}.$$

(a) Prove that A is dense in $\mathbb{R}^\omega$ with the product topology.
(b) Let $B \subset \mathbb{R}^\omega$ be the set of all bounded sequences. Prove that if $\mathbb{R}^\omega$ is given the box topology, then B is both open and closed.
(c) Conclude that A is not dense in $\mathbb{R}^\omega$ with the box topology.

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6. Let X be a compact metric space and suppose that $f\colon X \to X$ is an *isometry*: $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Prove that f is a homeomorphism. [Hint for surjectivity: if not, construct a sequence having no limit point.]