Exam III
Topology (Math 5853)
November 22, 2005

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1. Let X be the quotient space obtained from $\mathbb{C} \times \{0, 1\}$ by identifying $z \times 0$ with $z \times 1$ for every complex number z with $|z| > 1$.
(a) Does X satisfy the T_1 axiom? Why or why not?
(b) Is X Hausdorff? Why or why not?

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2. Prove the *Tube Lemma*: Consider the product space $X \times Y$ where Y is compact. If N is an open set of $X \times Y$ containing the subset $x_0 \times Y$, then x_0 has a neighborhood W in X such that $W \times Y$ is contained in N.

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3. Recall that a space X is *locally connected* if there is a basis for the topology of X consisting of connected sets.

(a) Give an example of a space that is locally connected but not connected (no proof required).

(b) Give an example of a space that is connected but not locally connected (no proof required).

(c) Prove that X is locally connected if and only if for every open set U of X, each component of U is open in X.

4(a) Define the *finite intersection property* and state (without proof) a definition of compactness in terms of this property.
(b) Prove the *contraction mapping theorem*: Let $f\colon X \to X$ be a continuous map where X is a compact metric space, and suppose there is a number $\lambda < 1$ such that

$$d(f(x), f(y)) \;\leq\; \lambda\, d(x, y)$$

for all $x, y \in X$. Then there is a unique point $x \in X$ such that $f(x) = x$.

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5. Consider the following subspace D of the plane. Let $K = \{1/n \mid n \in \mathbb{Z}_+\}$ and define

$$D = ([0, 1] \times 0) \cup (K \times [0, 1]) \cup (0 \times 1).$$

Let $p \in D$ be the point 0×1.

(a) Is D connected? Why or why not?
(b) Show that D is not path connected, as follows. Consider any path $f\colon [0, 1] \to D$ with $f(0) = p$. Show that $f(1)$ must also be p, by showing that $f^{-1}(\{p\}) = [0, 1]$. [Hint: show that $f^{-1}(\{p\})$ is open in $[0, 1]$. Then show that it is closed.]

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6. Let $p\colon X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected and Y is connected, then X is connected.
