Exam II
Topology (Math 5853)
October 25, 2005

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1. Let X be the set of real numbers with the finite complement topology (complements of finite sets are open).

(a) Does X satisfy the T_1 axiom?
(b) Is X Hausdorff?
(c) To what point or points does the sequence $x_n = 1/n$ converge?

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2. Consider the space $X = \mathbb{Z}_+ \times [0, 1)$ in the dictionary order topology. (The sets $\mathbb{Z}_+$ and $[0, 1)$ are given their usual orderings.) Construct a homeomorphism from X to the subspace $[0, \infty) \subset \mathbb{R}$ (and show that it is a homeomorphism).

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3. Let $f\colon A \to X \times Y$ be given by $f(a) = (f_1(a), f_2(a))$ where $f_1\colon A \to X$ and $f_2\colon A \to Y$ are functions. Show that if f_1 and f_2 are continuous then so is f.

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4. Let $f\colon \mathbb{R} \to \mathbb{R}^\omega$ be given by $f(t) = (t,\ \frac{1}{2}t,\ \frac{1}{4}t,\ \frac{1}{8}t,\ \dots)$.

(a) Show that f is continuous if $\mathbb{R}^\omega$ is given the product topology.
(b) Show that f is not continuous if $\mathbb{R}^\omega$ is given the box topology.

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5. Determine the closures of the following sets:

(a) $A = \{(1/n) \times 0 \mid n \in \mathbb{Z}_+\}$ in the ordered square ($I \times I$ in the dictionary topology)
(b) $K = \{1/n \mid n \in \mathbb{Z}_+\}$ in the set $\mathbb{R}$ with topology given by the basis $\{(-\infty, a) \mid a \in \mathbb{R}\}$
(c) $B = \{x \times \frac{1}{2} \mid 0 < x < 1\}$ in the ordered square

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6. Let a, b be points in a space Z. A *path* from a to b is a continuous map $f\colon [0, 1] \to Z$ such that $f(0) = a$ and $f(1) = b$. Consider the following subspace of $\mathbb{R}^2$:

$$Z = \{(x, y) \mid x \in \mathbb{Q},\ y > 0\} \ \cup\ \{(x, 0) \mid x \in \mathbb{R}\}.$$

Let $a = (x_a, y_a) \in Z$ and $b = (x_b, y_b) \in Z$, where $x_a < x_b$.

(a) Show that for any path in Z from a to b, its image contains the interval $(x_a, x_b) \times \{0\}$.
(b) Given arbitrary points $a, b \in Z$, describe the shortest path in Z from a to b, and write down an expression for its length. (This defines a metric on Z, different from the usual metric in $\mathbb{R}^2$.)
