Exam I

Topology (Math 5853)

September 27, 2005

1(a) State the axioms for $\mathscr{B}$ to be a basis.
(b) Define the topology $\mathscr{T}$ generated by $\mathscr{B}$.
(c) Show that if $\mathscr{B}$ is a basis for a topology on X, then the topology generated by $\mathscr{B}$ equals the intersection of all topologies on X that contain $\mathscr{B}$.

2(a) Recall the definition of $A + B$ for ordered sets A and B: $a < b$ for all $a \in A$ and $b \in B$, and the orderings within A and B are unchanged. Each of the following sets is given the dictionary order. Identify the order type of each one as $\mathbb{Z}_+$ or $\mathbb{Z}_+ + \mathbb{Z}_+$ or $\mathbb{Z}_+ + \mathbb{Z}_+ + \mathbb{Z}_+$, etc.

(i) $\{0,1\} \times \{0,1\} \times \mathbb{Z}_+$
(ii) $\{0,1\} \times \mathbb{Z}_+ \times \{0,1\}$
(iii) $\mathbb{Z}_+ \times \{0,1\} \times \{0,1\}$

(b) Show that $\mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+ \times \cdots$ in the dictionary order is not well-ordered.

---

3. Let X be an infinite set.
(a) Show that there is an injective map $f : \mathbb{Z}_+ \to X$.
(b) Show that for any $n \in \mathbb{Z}_+$, there is a bijection between X and X with n points removed.

---

4. Let X be a well-ordered set.
(a) Define the least upper bound property and the greatest lower bound property.
(b) Show that X has the least upper bound property.
(c) Show that X has the greatest lower bound property.

---

5. Consider the two injective maps $\mathbb{Z}_+ \to \mathbb{Z}_+$ given by $f(x) = x + 2$ and $g(x) = x + 3$. The proof of the Schroeder–Bernstein Theorem constructs a bijection $\mathbb{Z}_+ \to \mathbb{Z}_+$ based on f and g.
(a) Describe the *orbits* of the construction in this example. How many are there? (It may help to draw a picture.)
(b) Write down the bijection that the construction gives.

6(a) Show that $\mathscr{B} = \{(a,b) \mid a < b,\ a \text{ and } b \text{ rational}\}$ is a basis for the standard topology on $\mathbb{R}$.
(b) Recall that the lower limit topology has basis $\mathscr{B}_\ell = \{\,[a,b) \mid a < b\}$. Show that the basis $\mathscr{B}' = \{\,[a,b) \mid a < b,\ a \text{ and } b \text{ rational}\}$ generates a topology different from the lower limit topology.
