1. (a) Suppose $x_1A^1 + \cdots + x_nA^n = B$ where A^j and B are column vectors in K^n. Use properties of the determinant to simplify $D(A^1, \ldots, B, \ldots, A^n)$ where B appears in position j. Deduce (and state carefully) Cramer's Rule.

(b) Use Cramer's Rule to find x_1 and x_2 such that $x_1\begin{pmatrix}3\\4\end{pmatrix} + x_2\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\5\end{pmatrix}$.

(a) $$D(A^1, \ldots, \overset{j}{B}, \ldots, A^n) = D(A^1, \ldots, x_1A^1 + \cdots + x_nA^n, \ldots, A^n)$$

$$\underset{\text{(multilinearity)}}{=} x_1 D(A^1, \ldots, \overset{j}{A^1}, \ldots, A^n) + \cdots + x_n D(A^1, \ldots, \overset{j}{A^n}, \ldots, A^n)$$

$$\underset{\text{(alternating)}}{=} x_j D(A^1, \ldots, \overset{j}{A^j}, \ldots, A^n).$$

$$\boxed{\text{So (Cramer's Rule) if } D(A^1, \ldots, A^n) \neq 0 \text{ then } x_j = \frac{D(A^1, \ldots, B, \ldots, A^n)}{D(A^1, \ldots, A^j, \ldots, A)}.}$$

(b) $$x_1 = \frac{\begin{vmatrix} 0 & 2 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{-10}{-5} = \boxed{2}$$

$$x_2 = \frac{\begin{vmatrix} 3 & 0 \\ 4 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}} = \frac{15}{-5} = \boxed{-3}$$

check: $2\begin{pmatrix}3\\4\end{pmatrix} - 3\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\5\end{pmatrix}$ ✓

2. (a) Let W be the subspace of $\mathbb{R}^3$ generated by $(1,0,1)$ and $(2,-1,1)$. Let $\mathbb{R}^3$ be given the scalar product $\langle X,Y\rangle = x_1y_1 + 2x_2y_2 + x_3y_3$. Find an orthogonal basis of W with respect to this scalar product.

(b) Find a basis of the space of solutions to the equation $x_1 - x_2 + x_3 = 0$. [Hint: this equation can also be written $X \cdot A = 0$ where A is the column vector $(1,-1,1)$.]

(a) Apply Gram-Schmidt to basis $\{(1,0,1),(2,-1,1)\}$:

$$V_1 = (1,0,1)$$

$$V_2 = (2,-1,1) - \frac{\langle (2,-1,1),(1,0,1)\rangle}{\langle (1,0,1),(1,0,1)\rangle}(1,0,1)$$

$$= (2,-1,1) - \frac{2+0+1}{2}(1,0,1)$$

$$= \left(\tfrac{1}{2},-1,\tfrac{-1}{2}\right).$$ Orthog. Basis $\boxed{\left\{(1,0,1),\left(\tfrac{1}{2},-1,\tfrac{-1}{2}\right)\right\}}$.

(b) Solution space = orthogonal complement to $(1,-1,1)$ in $\mathbb{R}^3$ (a plane).

Need 2 indep. vectors $\perp$ to $(1,-1,1)$.

Try $(0,1,0),(0,0,1)$, make each orthog. to $(1,-1,1)$ by G-S :

$$(0,1,0) - \frac{(0,1,0)\cdot(1,-1,1)}{(1,-1,1)\cdot(1,-1,1)}(1,-1,1)$$

$$= (0,1,0) + \tfrac{1}{3}(1,-1,1) = \left(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{3}\right).$$

$$(0,0,1) - \frac{(0,0,1)\cdot(1,-1,1)}{(1,-1,1)\cdot(1,-1,1)}(1,-1,1)$$

$$= (0,0,1) - \tfrac{1}{3}(1,-1,1) = \left(\tfrac{-1}{3},\tfrac{1}{3},\tfrac{2}{3}\right).$$

Basis for solution space : $\boxed{\left\{\left(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{3}\right),\left(\tfrac{-1}{3},\tfrac{1}{3},\tfrac{2}{3}\right)\right\}}$.

3. Let V be the $\mathbb{R}$–vector space of polynomial functions $f\colon \mathbb{R} \to \mathbb{R}$ of degree at most 3, ie. functions of the form

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0, \qquad a_i \in \mathbb{R}.$$

(a) Determine a basis of V and the dimension of V.

(b) Let f be the derivative map on polynomials, ie. the map $P \mapsto \frac{d}{dx}P$. Let g be "multiplication by x," ie, the map $P(x) \mapsto xP(x)$. Compute $f \circ g$ and $g \circ f$ on the polynomial $P(x)$ given above. Say briefly why these compositions are linear maps from V to V.

(c) Find the matrices for $f \circ g$ and $g \circ f$ with respect to the basis you chose for V.

(a) $\boxed{\text{basis } \{1, x, x^2, x^3\}, \text{ dimension } 4}$

every $P(x)$ is uniquely expressed as a linear combination of these elements,

(b) $f \circ g(P) = \frac{d}{dx}(xP(x)) = \frac{d}{dx}(a_3x^4 + a_2x^3 + a_1x^2 + a_0x)$

$= \boxed{4a_3x^3 + 3a_2x^2 + 2a_1x + a_0}$

$g \circ f(P) = x\, P'(x)$

$= x(3a_3x^2 + 2a_2x + a_1)$

$= \boxed{3a_3x^3 + 2a_2x^2 + a_1x}$

both are polys of degree ≤ 3, so both map $V \to V$.

linear since $\frac{d}{dx}$ and "mult. by x" are both linear.

(c) $f \circ g(1) = 1$, $f \circ g(x) = 2x$, $f \circ g(x^2) = 3x^2$, $f \circ g(x^3) = 4x^3$

matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

$g \circ f(1) = 0$, $g \circ f(x) = x$, $g \circ f(x^2) = 2x^2$, $g \circ f(x^3) = 3x^3$

matrix $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

4. True or False. Indicate your answer clearly on the left side. You do not need to show your work.

F ____ Suppose the $m \times n$ matrix A represents the linear map $F\colon K^n \to K^m$. Then $\operatorname{rank} A = \dim \ker F$.

T ____ The identity map $\mathrm{id}\colon K^n \to K^n$ satisfies $M_{\mathcal{B}}^{\mathcal{B}}(\mathrm{id}) = I_n$ for any basis $\mathcal{B}$.

T ____ If $\dim V = n$ and W is a subspace of V then $\dim W + \dim W^{\perp} = \dim V$ with respect to any non-degenerate scalar product.

T ____ Every linear functional $f \in (K^n)^*$ is of the form $f(X) = A \cdot X$ for some $A \in K^n$.

F ____ The dimension of the space of solutions to the system of equations $\begin{array}{c} 4x + 7y - 3z = 0 \\ 2x - y + z = 0 \end{array}$ is two.

F ____ If A is symmetric (ie. $A = {}^tA$) then $\mathrm{Det}(A) = 0$.

T ____ Let $\mathcal{B}, \mathcal{B}'$ be two bases of V. If $F\colon V \to V$ is linear, then $M_{\mathcal{B}}^{\mathcal{B}}(F)$ is invertible if and only if $M_{\mathcal{B}'}^{\mathcal{B}'}(F)$ is invertible.

T ____ If $\langle \cdot, \cdot \rangle$ is a positive definite scalar product, then $|\langle v, w \rangle| \leq ||v|| \cdot ||w||$ for all v, w.