1. Let V be the $\mathbb{R}$–vector space of polynomial functions $f : \mathbb{R} \to \mathbb{R}$ of degree at most 10, i.e. functions of the form

$$f(x) = a_{10}x^{10} + a_9x^9 + \cdots + a_1x + a_0, \qquad a_i \in \mathbb{R}$$

(a) Determine a basis of V and the dimension of V

(b) Consider the map $L\colon V \to \mathbb{R}$ given by

$$f \mapsto \int_0^1 f(x)\,dx.$$

Show that L is linear and determine the dimension of the kernel of L.

(a) basis: $\boxed{\{1, x, x^2, \ldots, x^{10}\}}$ since every polynomial $f(x)$ as above is uniquely expressed as a linear combination of these terms. Dimension $= \boxed{11}$.

(b) (LM1):

$$L(f(x) + g(x)) = \int_0^1 (f(x) + g(x))dx$$

$$= \int_0^1 f(x)dx + \int_0^1 g(x)dx = L(f(x)) + L(g(x)). \checkmark$$

(LM2):

$$L(cf(x)) = \int_0^1 (cf(x))dx = c\int_0^1 f(x)dx = cL(f(x)). \checkmark$$

So $L: V \to \mathbb{R}$ is linear.

The image of L is a subspace of $\mathbb{R}$, so must be $\{0\}$ or $\mathbb{R}$. It is $\mathbb{R}$, because any number can be $\int_0^1 f(x)dx$ for some $f \in V$ ($f(x) = c$, for ex.)

So $\dim(\operatorname{Im} L) = 1$.

Rank-Nullity $\Rightarrow$ $\dim(\ker L) = \dim V - \dim(\operatorname{Im} L)$

$$= 11 - 1$$

$$= \boxed{10}.$$

2. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^4$ be the linear map such that $f(0,1) = (4,3,7,3)$ and $f(1,1) = (6,7,7,7)$. Find the matrix for F, i.e. the matrix A such that $F = L_A$.

A will be a 4x2 matrix, and its two columns are given by $f(1,0)$ and $f(0,1)$.

We have $f(0,1)$, but what is $f(1,0)$?

Write $(1,0) = (1,1) - (0,1)$

then $f(1,0) = f(1,1) - f(0,1)$ by linearity

$$= (6,7,7,7) - (4,3,7,3)$$
$$= (2,4,0,4).$$

So $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \\ 0 & 7 \\ 4 & 3 \end{bmatrix}.$

3. If $A = (a_{ij})$ is an $n \times n$ matrix, define the *trace* of A to be $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$. You may assume that this defines a linear map $\operatorname{tr}\colon \operatorname{Mat}_{n\times n}(K) \to K$.

(a) Show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for $A, B \in \operatorname{Mat}_{n\times n}(K)$.

(b) If B is invertible, show that $\operatorname{tr}(B^{-1}AB) = \operatorname{tr}(A)$.

(c) Prove that there are no matrices A, B such that $AB - BA = I_n$.

(a) Say $A = (a_{ij})$, $B = (b_{jk})$.

The ii-entry of AB is $\sum_{j=1}^{n} a_{ij} b_{ji}$

So $tr(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ji} a_{ij}$

$$\underset{\substack{\text{re-ordering}\\ \text{summation}}}{=} \sum_{j=1}^{n} \underbrace{\sum_{i=1}^{n} b_{ji} a_{ij}}_{jj\text{-entry of } BA}$$

$= tr(BA)$

(b) $tr(B^{-1}(AB)) \underset{\text{by (a)}}{=} tr((AB)B^{-1}) = tr(A(BB^{-1}))$

$= tr(AI) = tr(A)$,

(c) Since trace is linear,

$tr(AB - BA) = tr(AB) - tr(BA)$,

but this is 0 by (a).

However, $tr(I_n) = n$.

Since $0 \neq n$, $AB - BA$ cannot be I_n.

4. Recall that the *rank* of a linear map L is the dimension of the image of L, and the *nullity* is the dimension of the kernel of L. Suppose $F\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $G\colon \mathbb{R}^2 \to \mathbb{R}^3$ are linear maps.

(a) What are the possible ranks and nullities of F and G?

(b) Prove that $G \circ F$ is not invertible.

(a) Since $\dim \mathbb{R}^3 = 3$, the possibilities for F are:

- $\operatorname{rk} F = 0$, $\operatorname{null} F = 3$
- $\operatorname{rk} F = 1$, $\operatorname{null} F = 2$
- $\operatorname{rk} F = 2$, $\operatorname{null} F = 1$

note, $\operatorname{rk} F = 3$ is not possible because the image of F is a subspace of $\mathbb{R}^2$.

The possibilities for G are:

- $\operatorname{rk} G = 0$, $\operatorname{null} G = 2$
- $\operatorname{rk} G = 1$, $\operatorname{null} G = 1$
- $\operatorname{rk} G = 2$, $\operatorname{null} G = 0$

.

(b) Since $\operatorname{rk} G \leq 2$, G cannot be surjective (because $\dim \mathbb{R}^3 = 3$). Hence $G \circ F$ cannot be surjective, because $\operatorname{Im}(G \circ F)$ is contained in $\operatorname{Im}(G) \neq \mathbb{R}^3$. Hence $G \circ F$ is not bijection, and not invertible.

OR Since nullity of F is ≥ 1, F is not injective. Hence $G \circ F$ is not injective, because $\operatorname{Ker} F \; (\neq \{0\})$ is contained in $\operatorname{Ker}(G \circ F)$. Hence $G \circ F$ is not bijective, ...

5. True or False. Indicate your answer clearly on the left side. You do not need to show your work.

T If $F\colon V \to W$ is linear and surjective, and $\dim V = \dim W$, then F is invertible.

Ker $F = \{0\}$ by rank-nullity

F If $F\colon V \to W$ is linear then V is the direct sum of $\operatorname{Ker} F$ and $\operatorname{Im} F$.

Im F is in W, not V

F Every diagonal matrix is invertible. There may be zeros on the diagonal

T If $U \cap W = \{O\}$ for subspaces $U, W \subset V$ then $\dim U + \dim W \leqslant \dim V$.

choose bases for U, W; then their union is linearly indep. in V

T Every linear map $f\colon \mathbb{R} \to \mathbb{R}$ is of the form $f(x) = ax$.

In the standard basis, a linear map $\mathbb{R} \to \mathbb{R}$ has a 1×1 matrix, (a). The map is multiplication by a.