1. Find the determinants of the following matrices. State clearly any theorems or properties that you use.

(a) $\begin{pmatrix} 1 & 2 & 3 & 64 & 42 \\ 1 & 1 & 1 & 11 & 21 \\ 2 & 2 & 2 & 99 & 85 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 3 & 1 & 5 & 7 \end{pmatrix}$

(a) $\text{Det} = \text{Det}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \cdot \text{Det}\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ by theorem: $\text{Det}\left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right) = \text{Det}A \cdot \text{Det}D$

$= \text{Det}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \text{Det}\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ subtracting $2 \times$ row 2 from row 3 (does not change det.)

$= \boxed{0}$ by expanding along bottom row.

(b) $\begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 3 & 1 & 5 & 7 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 4 & 11 & 7 \end{pmatrix} \to \begin{pmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 10 & 5 \end{pmatrix}$

expand first column:

$\text{Det} = -1 \cdot \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 2 \\ 0 & 10 & 5 \end{vmatrix}$

expand first column

$= -1 \cdot \left(3 \begin{vmatrix} 1 & 2 \\ 10 & 5 \end{vmatrix} - 4 \begin{vmatrix} 2 & 1 \\ 10 & 5 \end{vmatrix} \right)$

$= -\left(3(-15) - 4(0) \right) = \boxed{45}$.

2. Let V be a vector space over K and let $A\colon V \to V$ be a linear map. Suppose that the characteristic polynomial of A factors as

$$P_A(t) \;=\; (t-\alpha_1)\cdots(t-\alpha_n)$$

where $\alpha_1, \ldots, \alpha_n$ are distinct numbers in K (i.e. $\alpha_i \neq \alpha_j$ for $i \neq j$). Show that V has a basis consisting of eigenvectors for A. [You may quote theorems from the course.]

λ is an eigenvalue $\Leftrightarrow$ $P_A(\lambda) = 0$.

Evidently the eigenvalues are the numbers $\alpha_1, \ldots, \alpha_n$. Note that $P_A(t)$ has degree n, so $\dim V = n$. Let $v_1, \ldots, v_n$ be eigenvectors for $\alpha_1, \ldots, \alpha_n$ respectively. Since these eigenvalues are all different, the vectors $v_1, \ldots, v_n$ are linearly independent (this was proved in class).

Since we have n indep. vectors in an n-dimensional space, they are a basis of V.

3. A matrix $A \in M_{n\times n}(K)$ is **symmetric** if ${}^tA = A$, and is **skew-symmetric** if ${}^tA = -A$.

(a) What is the dimension of $M_{4\times 4}(K)$?

(b) Determine the dimension of the subspace $M^{\mathrm{sym}}_{4\times 4}(K)$ of symmetric matrices.

(c) Determine the dimension of the subspace $M^{\mathrm{skew}}_{4\times 4}(K)$ of skew-symmetric matrices.

(d) Given any $A \in M_{n\times n}(K)$, show that $A + {}^tA$ is symmetric and $A - {}^tA$ is skew-symmetric.

(e) Show that $M_{n\times n}(K)$ is the direct sum of $M^{\mathrm{sym}}_{n\times n}(K)$ and $M^{\mathrm{skew}}_{n\times n}(K)$.

(f) Find a linear map $f\colon\ M_{n\times n}(K) \to M_{n\times n}(K)$ whose image is $M^{\mathrm{sym}}_{n\times n}(K)$ and whose kernel is $M^{\mathrm{skew}}_{n\times n}(K)$.

(a) Dimension is 16.

(b) $\begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix}$ is the general form.

Dimension $= 10$.

(c) $\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$ Dimension $= 6$.

(d) $${}^t(A + {}^tA) = {}^tA + {}^t({}^tA) = {}^tA + A$$
$$= A + {}^tA$$

So $A + {}^tA$ is symmetric.

$${}^t(A - {}^tA) = {}^tA - {}^t({}^tA) = {}^tA - A$$
$$= -(A - {}^tA)$$

So $A - {}^tA$ is skew-symmetric.

(**3** continued)

(e) we show (i) $M^{sym}_{n\times n}(K) \cap M^{skew}_{n\times n}(K) = \{0\}$

(ii) $M^{sym}_{n\times n}(K) + M^{skew}_{n\times n}(K) = M_{n\times n}(K)$.

(i) If A is symmetric and skew-symmetric then $A = {}^tA = -A$, so A is the zero matrix, hence (i).

(ii) Given any $A \in M_{n\times n}(K)$, we can write A as

$$A = \underbrace{\tfrac{1}{2}(A + {}^tA)}_{\in M^{sym}} + \underbrace{\tfrac{1}{2}(A - {}^tA)}_{\in M^{skew}}.$$

(f) Let $f(A) = A + {}^tA$.

- $A \in \ker(f) \Leftrightarrow A + {}^tA = 0, \Leftrightarrow {}^tA = -A$

  so $\ker(f) = M^{skew}_{n\times n}(K)$.

- we already know $A + {}^tA$ is in M^{sym}, so $\operatorname{Im}(f) \subset M^{sym}_{n\times n}(K)$. For other inclusion, if $A \in M^{sym}_{n\times n}(K)$, then $f(\tfrac{1}{2}A) =$

  $\tfrac{1}{2}(A + {}^tA) = \tfrac{1}{2}(A + A) = A$. So $\operatorname{Im}(f) = M^{sym}_{n\times n}(K)$.

4. Using the characteristic polynomial, prove that the matrix $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ has no real eigenvalues unless $A = \pm I$.

$$P_A(t) = \begin{vmatrix} t-\cos\theta & \sin\theta \\ -\sin\theta & t-\cos\theta \end{vmatrix}$$

$$= (t-\cos\theta)^2 + \sin^2\theta$$

$$= t^2 - 2\cos\theta\, t + \underbrace{\cos^2\theta + \sin^2\theta}_{=1}$$

roots:

$$\frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

these are real $\Longleftrightarrow 4\cos^2\theta - 4 \geq 0$

$\Longleftrightarrow \cos^2\theta \geq 1$

$\Longleftrightarrow \cos\theta = \pm 1$

$\Longleftrightarrow A = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

since $\sin\theta = 0$ when $\cos\theta = \pm 1$.

5. Let V_0, V_1, …, V_n be vector spaces over a field K. A sequence of linear maps

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} V_{n-1} \xrightarrow{f_n} V_n$$

is called **exact** if $\ker(f_i) = \mathrm{Im}(f_{i-1})$ for all $i = 1, \dots, n-1$.

(a) Show that

$$\{0\} \xrightarrow{0} U \xrightarrow{f} V$$

is exact if and only if f is injective.

(b) Show that

$$V \xrightarrow{g} W \xrightarrow{0} \{0\}$$

is exact if and only if g is surjective.

(c) Suppose

$$\{0\} \xrightarrow{0} U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{0} \{0\}$$

is exact. Show that $\dim V = \dim U + \dim W$.

(a) exactness holds $\iff \mathrm{Im}(0) = \ker(f)$

$\iff \{0\} = \ker(f)$

$\iff f$ is 1-1.

(b) exactness holds $\iff \mathrm{Im}(g) = \ker(0)$

$\iff \mathrm{Im}(g) = W$

$\iff g$ is onto.

(c) rank-nullity for f: $\dim \mathrm{Im}(f) + \dim \ker(f) = \dim U$

for g: $\dim \mathrm{Im}(g) + \dim \ker(g) = \dim V$

equal by exactness at V.

also, by exactness at U and W we have:

$\ker(f) = 0$, $\mathrm{Im}(g) = W$

Subtract second equation from first: $\dim V - \dim U = \dim W$.

6. Consider the linear map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$.

(a) Write down a basis for the row space of F.

(b) Write down a basis for the column space of F.

(c) Write down a basis for the kernel of F.

(d) Write down a basis for the image of F.

(e) What is the rank of F?

(a) row space is generated by $(2,0,1), (1,1,1), (2,2,2)$.
but these are not independent.
Basis: $\{(2,0,1), (1,1,1)\}$.

(b) column space has same dim. as row space, so it's enough to pick 2 indep. columns.
basis: $\{(2,1,2), (0,1,2)\}$.

(c) this is the orthogonal complement of the row space.
So we want a single vector $\perp$ to $(2,0,1)$ and $(1,1,1)$. Gram Schmidt works, but so does the cross product: $(2,0,1) \times (1,1,1) = (-1,-1,2)$.

Check: $(2,0,1)\cdot(-1,-1,2) = 0$ ✓
$(1,1,1)\cdot(-1,-1,2) = 0$ ✓

basis: $\{(-1,-1,2)\}$.

(d) $\mathrm{Im}(F)$ = col. space
so basis: $\{(2,1,2), (0,1,2)\}$

(e) $\mathrm{rank}(F) = 2$.

7. True or False. Indicate your answer clearly on the left side. You do not need to show your work.

T Every function from a basis of V to W defines a linear map $V \to W$.

F Every function from a generating set of V to W defines a linear map $V \to W$.

F Every multilinear, alternating map $K^n \times \cdots \times K^n \to K$ is equal to the determinant.

T The intersection of two subspaces is a subspace.

T The matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ has a real eigenvalue for any value of θ.

F A linear map takes a basis to a basis.

F The dot product on K^n is a positive definite scalar product, for any field K.

T The permutation $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}$ is odd.

T If $A \in M_{n\times n}(\mathbb{C})$ then there is a non-singular matrix B such that $B^{-1}AB$ is upper triangular.

T If $\dim V = n \geq 1$ and V has a positive definite scalar product, then V has an orthonormal basis.