Take-home Midterm
Math 4513-002 Capstone
Due March 15, 2013

Rules. You may refer to your class notes, homework and quizzes, and the lecture notes on random walks linked to from the course web page. You may not use other sources (books, online resources, etc) or discuss the exam with others. Please write out your answers neatly and in complete sentences. Feel free to ask me for a hint if you are stuck. The exam is due in class on Friday March 15.

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1. Consider a random walk on $\mathbb{Z}/n\mathbb{Z}$, instead of the integers $\mathbb{Z}$. In other words, walk along a circle of circumference n, taking a step left or right of length 1 at each turn. (So, n steps in the same direction gets you back where you started.) Label the starting point 0, and the other possible locations 1, 2, ..., $n-1$ in order around the circle. Equivalently, label the positions by the elements of $\mathbb{Z}/n\mathbb{Z}$.

It is easily shown that any given position will eventually be reached, with probability one. We are interested in the following question: if you start at 0, which location is most likely to be the *last* location reached by the walk? (Take a minute to make a guess.) More specifically, for each location k, we want the *probability* that it is the last one hit. Call this probability p_k.

(a) Explain why if $k = 1$ then this probability is the same as the ruin probability $r(n-2, 1)$. [Hint: consider a "companion" walk on $\mathbb{Z}$.] By symmetry, $k = -1$ gives the same probability.

(b) Compute the probability when $k = 2$. This is slightly harder, since now there are *two* ways this position could be reached last: the walk hits position 1 before position 3, or the walk hits position 3 before position 1.

(c) Can you find p_k for any k? Or, can you come up with some reasoning that lets you deduce it from earlier computations? What is the final story?

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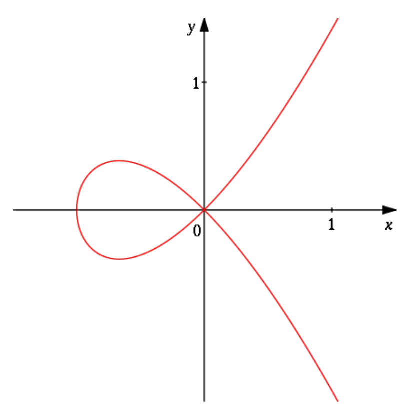
2. Let C be the cubic curve $y^2 = x^2(x+1)$, shown at the right. This is an example of a *singular* cubic. It turns out that singular cubics have a lot in common with conics.



(a) Show that the line through the origin with slope r meets the curve in exactly one point other than $(0, 0)$ (unless $r = \pm 1$). Find the coordinates of this point in terms of r.

(b) Find all the rational points on C. Explain why you found all of them.