1(a) Define the following:

(i) x is a limit point of A

(ii) A is closed

(iii) U is open

(b) Prove carefully that the set $(0, 1]$ is not open, and is not closed.

(a) (i) <u>x is a limit point of A</u> if: every neighborhood $V_\varepsilon(x)$ of x contains a point of A other than x.

(ii) <u>A is closed</u> if every limit point of A is in A

(iii) <u>U is open</u> if every $x \in U$ has a neighborhood $V_\varepsilon(x)$ which is contained in U.

(b) <u>$(0,1]$ is not open</u> since $x=1$ fails the requirement: $V_\varepsilon(x) = (1-\varepsilon, 1+\varepsilon)$ always contains a point (say, $1+\frac{\varepsilon}{2}$) not in $(0,1]$.

<u>$(0,1]$ is not closed</u> since 0 is a limit point. To see this, note that every neighborhood $V_\varepsilon(0) = (-\varepsilon, \varepsilon)$ contains a positive number (say, $\frac{\varepsilon}{2}$ if $\varepsilon < 2$) in $(0,1]$.

2(a) State the sequential condition for a function $f\colon A \to \mathbb{R}$ to be *not* uniformly continuous.

(b) Explain briefly why $f(x) = x^2$ is continuous.

(c) Show that $f(x) = x^2$ is not uniformly continuous.

(a) f is not uniformly continuous iff there exist sequences (x_n), (y_n) in A and $\varepsilon_0 > 0$ such that $|x_n - y_n| \to 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all n.

(b) $f(x)$ is the product $x \cdot x$, and $g(x) = x$ is continuous (given $\varepsilon > 0$ take $\delta = \varepsilon$; then $|x - c| < \delta \Rightarrow |x - c| < \varepsilon$) hence f is continuous.

(c) take $x_n = n$, $y_n = n + \frac{1}{n}$.

Then $|x_n - y_n| = |n - (n + \frac{1}{n})| = \frac{1}{n} \to 0$

and
$$|f(x_n) - f(y_n)| = |(n)^2 - (n + \tfrac{1}{n})^2| = |n^2 - n^2 - 2 - \tfrac{1}{n^2}| = 2 + \tfrac{1}{n^2} > 2.$$

So, taking $\varepsilon_0 = 2$, these sequences satisfy the criterion in (a). Hence $f(x)$ is not uniformly continuous.

3(a) Define the following:

(i) K is compact

(ii) E is connected

(b) Suppose K is compact and $f\colon K \to \mathbb{R}$ is continuous, and $f(x) > 0$ for all $x \in K$. Show that $g\colon K \to \mathbb{R}$, defined by $g(x) = 1/f(x)$, is bounded (ie. has image in $[-M, M]$ for some M).

(a) (i) K is compact if every sequence in K has a convergent subsequence, converging to a point in K.

(ii) E is connected if, whenever E is partitioned into two disjoint non-empty sets, one of the sets intersects the closure of the other.

(b) First proof f achieves a minimum at some $x_0 \in K$, since f is cont. and K is compact. That is, $f(x_0) \leq f(x)$ $\forall x \in K$.

Now, $g(x_0) \geq g(x)$ $\forall x \in K$ ($\frac{1}{x}$ is order reversing).

Since $g(x) \geq 0$ $\forall x \in K$ also, we now have

$g(K) \subset [0, g(x_0)] \subset [-g(x_0), g(x_0)]$.

Second Proof Since $f(x) \neq 0$ on K, g is continuous on K by the algebraic continuity theorem. Hence $g(K)$ is compact. By Heine-Borel, $g(K)$ is bounded.

4. Do the following items exist? Give examples (with brief explanations) or say why not.

(i) A function which is bounded but not uniformly continuous

(ii) A finite set that is not compact

(iii) A function which is continuous exactly on the set $(-\infty, 0] \cup (1, \infty)$

(iv) A non-empty set with no limit points

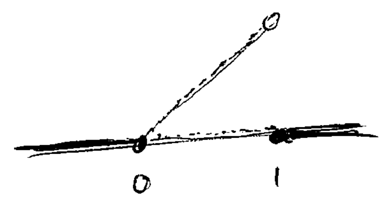
(v) A countable infinite compact set

(vi) A countable infinite connected set

(i) $\sin(\frac{1}{x})$ works. So does any bounded, non-continuous function.

(ii) Every finite set is compact, since it is closed and bounded.

(iii) $$f(x) = \begin{cases} 0 & x \leq 0 \text{ or } x \geq 1 \\ 0 & x \in (0,1) \cap \mathbb{Q} \\ x & x \in (0,1) - \mathbb{Q} \end{cases}$$



(iv) any non-empty finite set works

(v) $\{ \frac{1}{n} \mid n \in \mathbb{N} \} \cup \{0\}$

(vi) does not exist; if E is connected and $a, b \in E$ then the interval (a,b) is in E. Hence E is uncountable.