1(a) Define the following:

(i) x is an upper bound for A

(ii) x is a supremum of A

(iii) the Axiom of Completeness for $\mathbb{R}$

(b) Prove that $\sup A$, if it exists, is unique.

(a)(i) x is an upper bound for A if $x \geq a$ for every $a \in A$.

(ii) x is a supremum of A if (1) x is an upper bound for A, and (2) whenever b is an upper bound for A, we have $x \leq b$.

(iii) Every nonempty subset of $\mathbb{R}$ which has an upper bound also has a supremum.

(b) suppose s, t are supremums of A.
Then they are both upper bounds of A.
Property (2) for s then gives $s \leq t$.
Property (2) for t gives $t \leq s$.

So $s = t$.

2(a) Define: (x_n) is a Cauchy sequence.

(b) Prove that Cauchy sequences are bounded.

(a) (x_n) is a Cauchy sequence if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ s.t. for all $n, m \geq N$, $|x_n - x_m| < \varepsilon$.

(b) Taking $\varepsilon = 1$, there is an N s.t. $|x_n - x_m| < 1$ for all $n, m \geq N$. In particular, $|x_n - x_N| < 1$ $\forall n \geq N$.

This implies $|x_n| \leq |x_N| + 1$ for all $n \geq N$.

Let $M = \max\{|x_1|, |x_2|, \cdots, |x_{N-1}|, |x_N| + 1\}$.

Now $|x_n| \leq M$ for all n:

- if $n \geq N$ then this was shown above, since $|x_n| \leq |x_N| + 1 \leq M$
- if $n < N$ then $|x_n| \leq M$ by the def. of M.

So (x_n) is contained in $[-M, M]$.

3(a) Define: (x_n) converges to x.

(b) Suppose that $(x_n) \to x$ and $(y_n) \to y$. Let (z_n) be the sequence $(x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$.

(i) Prove that if $x = y$ then (z_n) converges.

(ii) Prove that if $x \neq y$ then (z_n) is not a Cauchy sequence. What can you then conclude?

(a) $(x_n) \to x$ if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq N$.

(b) (i) Given $\varepsilon > 0$,

there is an N_1 s.t. $|x_n - x| < \varepsilon$ for all $n \geq N_1$

and an N_2 s.t. $|y_n - y| < \varepsilon$ for all $n \geq N_2$.

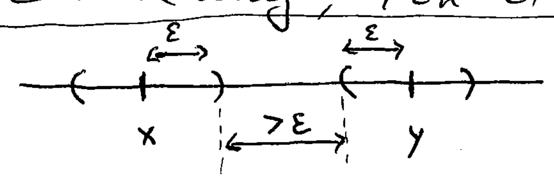
Note that $z_{2k} = y_k$, $z_{2k-1} = x_k$ for all $k \in \mathbb{N}$.

Let $N = \max\{2N_1, 2N_2\}$, let $z = x = y$.

- if $n = 2k$ and $n \geq N$ then $|z_n - z| = |y_k - y| < \varepsilon$ because $k \geq N_2$.
- if $n = 2k-1$ and $n \geq N$ then $|z_n - z| = |x_k - x| < \varepsilon$ because $k \geq N_1$.

either way, $|z_n - z| < \varepsilon$ for $n \geq N$. Hence $(z_n) \to z$.

(ii)



Pick $\varepsilon > 0$ with $3\varepsilon < |x-y|$ ⊛. Since $(x_n) \to x$ and $(y_n) \to y$, there is an N s.t. $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ for all $n \geq N$.

For such n, $|x_n - y_n| > \varepsilon$: (see picture)

$|x-y| \leq |x - x_n| + |x_n - y_n| + |y_n - y| < 2\varepsilon + |x_n - y_n|$

and so $|x_n - y_n| > |x-y| - 2\varepsilon > \varepsilon$ by ⊛.

Hence, $\forall M \in \mathbb{N}$, there exist $m, n \geq M$ with $|z_m - z_n| \geq \varepsilon$;

take $m = 2k-1$, $n = 2k$ where $k \geq N$ (and $2k-1 \geq M$),

then $|z_m - z_n| = |x_k - y_k| > \varepsilon$. Hence (z_n) is not Cauchy, and hence (z_n) does not converge.

4. Do the following items exist? Give examples (with brief explanations) or say why not.

(i) A convergent series $\sum_k a_k$ such that $\sum_k |a_k|$ diverges

(ii) A divergent series $\sum_k a_k$ such that $\sum_k |a_k|$ converges

(iii) A monotone sequence with no convergent subsequence

(iv) A sequence with no terms equal to natural numbers, but with subsequences converging to two different natural numbers.

(v) A sequence ~~with no terms equal to natural numbers, but~~ with subsequences converging to all natural numbers.

(vi) Sequences $(a_n) \to a$ and $(b_n) \to b$ such that $a_n > b_n$ for all n and $a \not> b$.

(vii) Sequences $(a_n) \to a$ and $(b_n) \to b$ such that $a_n > b_n$ for all n and $a < b$.

(i) $\sum_k \frac{(-1)^{k+1}}{k}$ converges by alt. series test; $\sum_k \left|\frac{(-1)^{k+1}}{k}\right|$ is the harmonic series (diverges)

(ii) does not exist, by abs. conv test

(iii) (a_n) where $a_n = n$; every subsequence is unbounded, hence does not converge

(iv) $a_n = (-1)^n + \frac{1}{n+1}$

then even terms are $\left(1 + \frac{1}{2n+1}\right) \to 1$

odd terms are $\left(-1 + \frac{1}{2n+2}\right) \to -1$

(v) $(1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 6, \cdots)$

for every n there is a subsequence of the form $(n, n, n, \dots)$, which converges to n.

(v)′ add $\left(\frac{1}{n+1}\right)$ to the above sequence. This does not change limits of subsequences (b/c it $\to 0$).

(vi) $(a_n) = \left(\frac{1}{n}\right)$, $(b_n) = (0)$ both have limit 0.

(vii) Impossible; by limit order theorem, if $a_n > b_n$ then then $a \geq b$.