1(a) Define: x is a least upper bound for A.

(b) State the Axiom of Completeness for $\mathbb{R}$.

(c) Suppose $I_n = [a_n, b_n]$ are closed intervals such that $I_{n+1} \subset I_n$ for all $n \in \mathbb{N}$. Show that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

[Hint: use the sequence (a_n).]

(a) x is an l.u.b. for A if:

(i) $x \geq a$ for all $a \in A$ (x is an upper bound for A)

(ii) whenever y is an upper bound for A, $x \leq y$

(b) every non-empty set which has an upper bound also has a least upper bound.

(c) First, $a_n \leq b_m$ for all n, m:

- if $n \leq m$ then $a_n \leq a_m < b_m$
- if $n \geq m$ then $a_n < b_n \leq b_m$.

So, every b_m is an upper bound for $\{a_n \mid n \in \mathbb{N}\}$.

Hence this set has a least upper bound, x.

By (ii) above, $x \leq b_m$ for all m.

By (i) above, $x \geq a_n$ for all n.

Hence $x \in [a_n, b_n]$ for all n

$\Rightarrow x \in \bigcap_{n \in \mathbb{N}} I_n$.

2(a) Define: x is a limit point of A.

(b) Show that if y is a limit point of $A \cup B$ then y is either a limit point of A or a limit point of B (or both).

(c) Conclude that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Is it true in general that $\overline{\bigcup A_i} = \bigcup \overline{A}_i$ for infinite unions?

(a) every neighborhood $V_\varepsilon(x)$ intersects A in a point $\neq x$.

(b) if not, there exist neighborhoods $V_{\varepsilon_1}(y)$ and $V_{\varepsilon_2}(y)$ such that $V_{\varepsilon_1}(y) \cap (A - \{y\}) = \emptyset$ and $V_{\varepsilon_2}(y) \cap (B - \{y\}) = \emptyset$.

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $V_\varepsilon(y) \cap ((A \cup B) - \{y\})$

$\subset \left(V_\varepsilon(y) \cap (A - \{y\})\right) \cup \left(V_\varepsilon(y) \cap (B - \{y\})\right) = \emptyset \cup \emptyset = \emptyset$

So y is not a limit point of $A \cup B$.

(c) $\overline{A \cup B} = (A \cup B) \cup \{\text{limit pts of } A \cup B\}$

$= A \cup B \cup \{\text{limit pts of } A\} \cup \{\text{limit pts of } B\}$ by (b)

$= \overline{A} \cup \overline{B}$.

In general, can have $\overline{\bigcup A_i} \neq \bigcup \overline{A}_i$ e.g.

take $A_i = [\frac{1}{i}, 1]$, so $A_i = \overline{A_i}$ and

$\bigcup \overline{A}_i = \bigcup A_i = (0, 1]$

$\overline{\bigcup A_i} = \overline{(0,1]} = [0, 1]$.

3(a) Define *uniform continuity* of f on A.

(b) Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, \infty)$.

(a) f is <u>uniformly continuous on A</u> if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. for every $x, y \in A$: $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

(b) Given $\varepsilon > 0$, and $x, y \geq 1$:

$$|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{y^2 - x^2}{x^2 y^2} \right| = \frac{|y-x||y+x|}{x^2 y^2}$$

Consider $\frac{|y+x|}{x^2y^2} = \frac{y}{x^2y^2} + \frac{x}{x^2y^2}$ (x, y are positive)

$$= \frac{1}{x^2 y} + \frac{1}{x y^2}.$$

Since $x, y \geq 1$, this is $\leq 1 + 1 = 2$.

Take $\delta = \varepsilon/2$.

Now if $|x-y| < \delta$, we have

$$|f(x) - f(y)| \leq |y-x| \cdot 2$$

$$< 2\delta = \varepsilon.$$

4(a) Given $f\colon A \to \mathbb{R}$ and $c \in A$, define $f'(c)$.

(b) State the Mean Value Theorem.

(c) Recall that $f\colon (a,b) \to \mathbb{R}$ is *increasing* if $f(x) \le f(y)$ whenever $x < y$ in (a,b). Suppose f is differentiable on (a,b). Show that $f'(c) \ge 0$ for all $c \in (a,b)$ if and only if f is increasing.

(a) $f'(c)$ is $\displaystyle\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ if this limit exists.

(b) If f is continuous on $[a,b]$ and differentiable on (a,b) then $\exists\, c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

(c) $\underline{\Rightarrow}$: Suppose $f'(c) \geq 0$ $\forall c \in (a,b)$.

If $x,y \in (a,b)$ with $x<y$, apply MVT to f on $[x,y]$. Get $c \in (x,y)$ with $f'(c) = \dfrac{f(y)-f(x)}{y-x}$. Since $f'(c) \geq 0$, and $y-x > 0$, we have $f(y)-f(x) \geq 0$, i.e. $f(x) \le f(y)$.

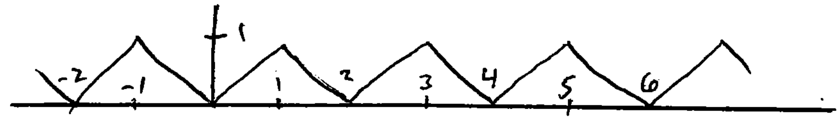
$\underline{\Leftarrow}$: Given $c \in (a,b)$, let $L = f'(c)$. Using one-sided limits, we have $L = \displaystyle\lim_{x\to c^+} \frac{f(x)-f(c)}{x-c}$ (since the limit exists). In this limit, we have $x > c$, and since f is increasing, $f(x) \geq f(c)$. So the difference quotients are all ≥ 0. Hence $L \geq 0$.

5. Let (f_n) be a sequence of functions defined on A.

(a) State the Weierstrass M-test for uniform convergence of $\sum f_n$.

(b) Let $h(x)$ be the sawtooth function given by: $|x|$ on $[-1,1]$, and by $|x-2n|$ on $[2n-1, 2n+1]$. Let $f_n(x) = \frac{1}{2^n}h(2^n x)$ for each n (the graph is just like that of $h(x)$, but scaled down by a factor of 2^n in all directions).

h:



Use the M-test to prove that $g(x) = \sum \frac{1}{2^n}h(2^n x)$ is uniformly convergent. What can you then conclude about $g(x)$, using other results?

(a) if there exist numbers $M_n \geq 0$ s.t. $|f_n(x)| \leq M_n$ for all $x \in A$, for all n, and if $\sum_{n\in\mathbb{N}} M_n$ converges, then $\sum_{n\in\mathbb{N}} f_n$ converges uniformly on A.

(b) Note that $f_n(x) \in [0, \frac{1}{2^n}]$ for all x so $|f_n| \leq \frac{1}{2^n}$. Also, $\sum \frac{1}{2^n}$ converges since it is a geometric series with $r = \frac{1}{2}$.
Hence $g(x) = \sum f_n(x)$ converges uniformly on $\mathbb{R}$.

Moreover, since each f_n is continuous and $\sum f_n$ converges uniformly, $g(x)$ is continuous on $\mathbb{R}$ (by problem 6(b)).

6. Let (f_n) be a sequence of functions defined on A.

(a) Define: the sequence (f_n) converges uniformly to f on A.

(b) Prove that if each f_n is continuous at c and $(f_n) \to f$ uniformly on A, then f is continuous at c.

(a) $(f_n) \to f$ uniformly on A if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$
s.t. $\forall x \in A \quad n \geq N \Rightarrow |f_n(x) - f(x)| < \varepsilon$.

(b) for any n and any $x \in A$,

$|f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)|$
by triangle inequality.

Given $\varepsilon > 0$,
Pick N (by uniform convergence) s.t. $\forall n \geq N$
$\forall z \in A, \quad |f_n(z) - f(z)| < \varepsilon/3$.
In particular, $|f_N(x) - f(x)| < \varepsilon/3$ and $|f_N(c) - f(c)| < \varepsilon/3$.
Now, since f_N is continuous at c, there is a $\delta > 0$
s.t. $|x - c| < \delta \rightarrow |f_N(x) - f_N(c)| < \varepsilon/3$.

For this δ, we now have

$|x - c| < \delta \rightarrow |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)|$
$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

So $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

7(a) Do the following items exist? Give examples (with brief explanations) or say why not.

(i) a sequence (b_n) and a number b such that $|b_n| \to |b|$ and $b_n \not\to b$

(ii) a bounded sequence such that no subsequence is a Cauchy sequence

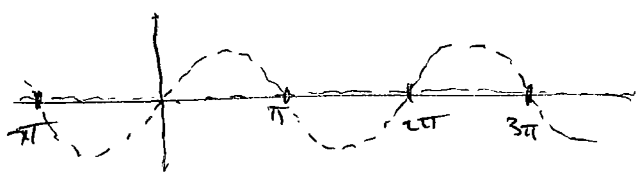
(iii) a function which is continuous exactly at the points $n\pi$, for all $n \in \mathbb{Z}$

(iv) a continuous function $f\colon (0,1) \to \mathbb{R}$ and a Cauchy sequence (x_n) in $(0,1)$ such that $(f(x_n))$ is not a Cauchy sequence

(i) take $b_n = (-1)^n$, $b = 1$ then $|b_n| = 1 \to 1$ ✓
but (b_n) does not converge to anything

(ii) every bounded sequence has a convergent subsequence, which will be Cauchy (b/c it converges)

(iii) $f(x) = \begin{cases} \sin(x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$



(iv) $f(x) = \frac{1}{x}$, $(x_n) = \frac{1}{n}$ ← Cauchy b/c it converges to 0
but $(f(x_n)) = (n)$ which diverges

7(b) True or False (no explanation needed):

T — if K_n are compact sets such that $K_{n+1} \subset K_n$ for all n then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$

F — the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ can be rearranged to sum to π absolutely convergent

T — if a power series $\sum_{n=0}^{\infty} a_n x^n$ converges pointwise on $[-2,2]$ then it is continuous on $[-2,2]$

F — if a power series $\sum_{n=0}^{\infty} a_n x^n$ converges pointwise on $[-2,2]$ then it is differentiable on $[-2,2]$