1. (6 points) Use an integrating factor to solve the linear differential equation $3xy' + y = 12x$.

$$y' + \underbrace{\frac{1}{3x}}_{P(x)} y = \underbrace{4}_{Q(x)} \qquad \rho(x) = e^{\int \frac{1}{3x}\,dx} = e^{\frac{1}{3}\ln x} = x^{1/3}.$$

$$x^{1/3} y' + x^{1/3} \frac{1}{3x} y = 4x^{1/3}.$$

$$\frac{d}{dx}\left(x^{1/3} y\right) = 4x^{1/3}$$

$$x^{1/3} y = \int 4x^{1/3}\,dx = 3x^{4/3} + C$$

$$\boxed{y = 3x + \frac{C}{x^{1/3}}}$$

2. (6 points) Use the substitution $v = y^3$ to transform the first order equation $3xy^2y' = 3x^4 + y^3$ into a linear differential equation. Then find the integrating factor for the linear equation, but do not proceed any further.

$$v = y^3, \quad y = v^{1/3}, \quad \frac{dy}{dx} = \frac{1}{3} v^{-2/3} \frac{dv}{dx}$$

substitute:

$$3x v^{2/3} \frac{1}{3} v^{-2/3} \frac{dv}{dx} = 3x^4 + v$$

$$x \frac{dv}{dx} = 3x^4 + v$$

$$\rho(x) = e^{\int \frac{-1}{x}\,dx} = e^{-\ln x} = x^{-1}$$

$$\boxed{\frac{dv}{dx} - \frac{1}{x} v = 3x^3}$$

$$\boxed{\rho(x) = \frac{1}{x}}$$

**3.** (5 points each) Calculate the following Laplace transforms and inverse Laplace transforms, following any special instructions given. Make use of the table of formulas whenever possible.

(a) $\mathcal{L}\{f(t)\}$ where $f'(t) = \sinh(5t)$ and $f(0) = 4$, using the formula for $\mathcal{L}\{f'(t)\}$.

$$\mathcal{L}\{f'\} = s\,\mathcal{L}\{f\} - f(0)$$

$$\frac{5}{s^2-25} = s\,\mathcal{L}\{f\} - 4$$

$$\boxed{\mathcal{L}\{f\} = \frac{1}{s}\left(\frac{5}{s^2-25} + 4\right)}$$

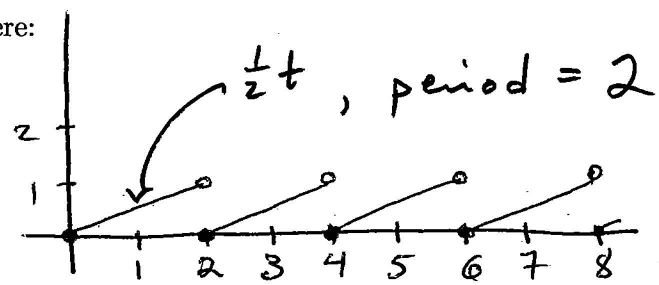
(b) $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 0 & t < \pi \\ \sin t & t \geq \pi. \end{cases}$ (Hint: write $f(t)$ using a step function.)

$$f(t) = u(t-\pi)\sin t$$

$$= -u(t-\pi)\sin(t-\pi)$$

$$\boxed{\mathcal{L}\{f\} = e^{-\pi s}\left(\frac{-1}{s^2+1}\right)}$$

(c) $\mathcal{L}\{f(t)\}$, where $f(t)$ is the function shown here:




$$\mathcal{L}\{f\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}\left(\tfrac{1}{2}t\right)dt$$

$$\left[\begin{array}{ll} u = t & v = \frac{-1}{s}e^{-st} \\ du = dt & dv = e^{-st}dt \end{array}\right]$$

$$= \frac{1}{2(1-e^{-2s})}\left(\left[\frac{-1}{s}te^{-st}\right]_0^2 - \int_0^2 \frac{-1}{s}e^{-st}dt\right)$$

$$= \boxed{\frac{1}{2(1-e^{-2s})}\left(\frac{-2}{s}e^{-2s} - \frac{1}{s^2}e^{-2s} + \frac{1}{s^2}\right)}$$

**(d)** $\mathcal{L}^{-1}\{\tan^{-1}(1/s)\}$, using the fact that $\frac{d}{ds}\left(\tan^{-1}(1/s)\right) = \frac{-1}{s^2+1}$.

$$-t f(t) = \mathcal{L}^{-1}\left\{\frac{d}{ds} F(s)\right\}$$

$$-t f(t) = \mathcal{L}^{-1}\left\{\frac{-1}{s^2+1}\right\} = -\sin t$$

$$\boxed{f(t) = \frac{\sin t}{t}}$$

**(e)** $\mathcal{L}\left\{\frac{e^t - e^{-t}}{t}\right\}$ $\quad \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$

$$\mathcal{L}\left\{\frac{e^t - e^{-t}}{t}\right\} = \int_s^\infty \left(\frac{1}{\sigma-1} - \frac{1}{\sigma+1}\right) d\sigma$$

$$= \Big[\ln|\sigma-1| - \ln|\sigma+1|\Big]_s^\infty$$

$$= \left[\ln\left|\frac{\sigma-1}{\sigma+1}\right|\right]_s^\infty = \boxed{-\ln\left|\frac{s-1}{s+1}\right|}$$

**(f)** $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\cdot\frac{1}{s-b}\right\}$, where $a \neq b$, using the convolution formula. Simplify your answer to the point where there is no integration symbol.

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\cdot\frac{1}{s-b}\right\} = e^{at} * e^{bt} = \int_0^t e^{a\tau} e^{b(t-\tau)}\,d\tau$$

$$= e^{bt}\int_0^t e^{(a-b)\tau}\,d\tau = e^{bt}\left[\frac{1}{a-b} e^{(a-b)\tau}\right]_0^t$$

$$= e^{bt}\left[\frac{1}{a-b} e^{(a-b)t} - \frac{1}{a-b}\right]$$

$$= \boxed{\frac{e^{at} - e^{bt}}{a-b}}$$

**4a.** (3 points) Use the characteristic equation to find a general solution to $y'' + 6y' + 9y = 0$.

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0 \qquad r = -3, -3$

$$\boxed{y = Ae^{-3x} + Bxe^{-3x}}$$

**4b.** (5 points) Given that $y = e^{2x}$ is a solution to $y'' + 6y' + 9y = 25e^{2x}$, solve the initial value problem $y'' + 6y' + 9y = 25e^{2x}$, $y(0) = 2$, $y'(0) = 2$.

$y = y_c + y_p = Ae^{-3x} + Bxe^{-3x} + e^{2x}$

$y(0) = 2: \quad 2 = A + 1, \quad \underline{A = 1}$

$y' = -3Ae^{-3x} - 3Bxe^{-3x} + Be^{-3x} + 2e^{2x}$

$y'(0) = 2: \quad 2 = -3A + B + 2$

$= -3 + B + 2, \quad \underline{B = 3}$

$$\boxed{y = e^{-3x} + 3xe^{-3x} + e^{2x}}$$

**5.** (5 points) Find all solutions to the separable differential equation $(1+x)\frac{dy}{dx} = 4y$. Be sure to include any singular solutions.

$\int \frac{dy}{4y} = \int \frac{dx}{1+x}$

$\frac{1}{4} \ln|y| = \ln|1+x| + C$

$\ln|y| = 4\ln|1+x| + 4C$

$y = \pm e^{4\ln|1+x|} e^{4C}$

$y = \pm(1+x)^4 e^{4C}$

also $y = 0$, so

$$\boxed{y = A(1+x)^4}$$

**6.** (6 points) Consider the endpoint problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi) = 0$. Is $\lambda = 4$ an eigenvalue? If so give an eigenfunction; otherwise say why not.

$y'' + 4y = 0 \qquad r^2 + 4 = 0, \qquad r = \pm 2i$

$y = A\cos(2x) + B\sin(2x)$

$y(0) = 0: \quad 0 = A + B(0), \quad A = 0.$

$y = B\sin(2x)$

$y(\pi) = 0: \quad 0 = B\sin(2\pi) = B(0).$

$B$ unconstrained $\Rightarrow$ $y = B\sin(2x)$ is a solution.

So 4 is an eigenvalue;
$y = B\sin(2x)$ is an eigenfunction (for any $B \neq 0$)

**7.** (6 points) Use the Laplace transform method to solve the following initial value problem. Write your answer without step functions (i.e. use cases if necessary), and simplify it as much as you can.

$$y'' + y = \delta(t - \pi), \quad y(0) = 0, \quad y'(0) = 1$$

$(s^2 Y(s) - 1) + Y(s) = e^{-\pi s}$

$Y(s) = \dfrac{e^{-\pi s} + 1}{s^2 + 1} = e^{-\pi s}\dfrac{1}{s^2+1} + \dfrac{1}{s^2+1}$

$y(t) = u(t-\pi)\sin(t-\pi) + \sin t$

$= \begin{cases} \sin t & t < \pi \\ \sin(t-\pi) + \sin t & t \geq \pi \end{cases}$

$$\boxed{y(t) = \begin{cases} \sin t & t < \pi \\ 0 & t \geq \pi \end{cases}}$$

Use the power series method to solve the differential equation $y' + 4y = 0$, as follows.

**8a.** (4 points) Substitute $y = \sum_{n=0}^{\infty} c_n x^n$ into the equation and find a recurrence relation satisfied by the coefficients $c_n$.

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 4 \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left[ (n+1) c_{n+1} + 4 c_n \right] x^n = 0$$

$(n+1) c_{n+1} + 4 c_n = 0$, all $n$.

$$\boxed{c_{n+1} = \frac{-4 c_n}{n+1}}$$

**8b.** (4 points) Find $c_1$, $c_2$, $c_3$, etc. in terms of $c_0$, and write $y$ as a series. Then express the general solution $y$ using a closed formula (i.e. a formula with no summation symbol).

$$c_1 = \frac{-4 c_0}{1}$$

$$c_2 = \frac{-4 c_1}{2} = \frac{4 \cdot 4 c_0}{2}$$

$$c_3 = \frac{-4 c_2}{3} = \frac{-4 \cdot 4 \cdot 4 c_0}{3 \cdot 2}$$

$$c_4 = \frac{-4 c_3}{4} = \frac{4 \cdot 4 \cdot 4 \cdot 4 c_0}{4 \cdot 3 \cdot 2}$$

$$\vdots$$

$$\boxed{c_n = \frac{(-4)^n c_0}{n!}}$$

$$\boxed{y = \sum_{n=0}^{\infty} \frac{(-4)^n c_0}{n!} x^n}$$

$$= c_0 \sum_{n=0}^{\infty} \frac{1}{n!} (-4x)^n$$

$$\boxed{y = c_0 e^{-4x}}$$

**Bonus.** (2 points) Find the radius of convergence of the series for the previous problem (and give justification).

$$\text{radius} = \lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(-4)^n c_0}{n!} \cdot \frac{(n+1)!}{(-4)^{n+1} c_0} \right|$$

$$= \lim_{n \to \infty} \left| \frac{n+1}{-4} \right| \quad \boxed{= \infty, \text{ converges for all } x}.$$

| Function | Laplace Transform |
|---|---|
| $f(t)$ | $F(s)$ |
| $af(t)+bg(t)$ | $aF(s)+bG(s)$ |
| $f'(t)$ | $sF(s)-f(0)$ |
| $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$ |
| $\int_0^t f(\tau)d\tau$ | $\frac{F(s)}{s}$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $u(t-a)f(t-a)$ | $e^{-as}F(s)$ |
| $(f*g)(t)$ | $F(s)G(s)$ where $(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$ |
| $tf(t)$ | $-F'(s)$ |
| $t^nf(t)$ | $(-1)^nF^{(n)}(s)$ |
| $\frac{f(t)}{t}$ | $\int_s^\infty F(\sigma)d\sigma$ |
| $f(t)$, period $p$ | $\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $t^a$ | $\frac{\Gamma(a+1)}{s^{n+1}}$ where $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\cos(kt)$ | $\frac{s}{s^2+k^2}$ |
| $\sin(kt)$ | $\frac{k}{s^2+k^2}$ |
| $\cosh(kt)$ | $\frac{s}{s^2-k^2}$ |
| $\sinh(kt)$ | $\frac{k}{s^2-k^2}$ |
| $u(t-a)$ | $\frac{e^{-as}}{s}$ where $u(t-a)$ is 0 when $t<a$, and 1 when $t\geq a$ |
| $\delta(t-a)$ | $e^{-as}$ where $\delta(t-a)$ is a unit impulse at time $t=a$ |

## Power Series

$\cos x = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}x^{2n}$

$\sin x = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

$\cosh x = \sum_{n=0}^\infty \frac{1}{(2n)!}x^{2n}$

$\sinh x = \sum_{n=0}^\infty \frac{1}{(2n+1)!}x^{2n+1}$

$e^x = \sum_{n=0}^\infty \frac{1}{n!}x^n$

$\ln(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}x^n$

$\frac{1}{1-x} = \sum_{n=0}^\infty x^n$