1. (5) Transform the following system into an equivalent first order system of differential equations:

$$2x'' + x + y' - x^{(3)} = \cos t$$

$$y + 2y' + x'' - 10y'' = 0$$

$x_1 = x'$
$x_2 = x_1'$
$y_1 = y'$
$2x_2 + x + y_1 - x_2' = \cos t$
$y + 2y_1 + x_2 - 10y_1' = 0$

2. (7) Use the method of elimination to transform the following system into a single differential equation involving t, x, x', x'', etc. (with no y). Do not solve the equation, but write it in the usual way, without operators.

$$3x' = y - 2x - y' + \sin t$$

$$6x + y' = y$$

$(3D+2)x + (D-1)y = \sin t$

$-\quad 6x + (D-1)y = 0$

$(3D+2)x - 6x = \sin t$

$3x' - 4x = \sin t$

3a. (4) Find the operational determinant of the following system:

$$D^2x + Dx + D^4y = 0$$

$$Dx + D^3y + Dy = 0$$

$(D^2+D)x + D^4 y = 0$

$Dx + (D^3+D)y = 0$

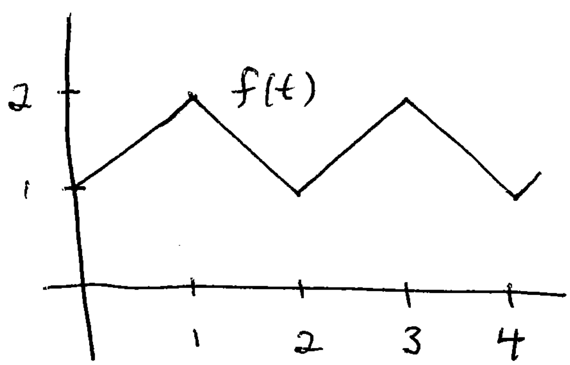
Op. det. $= (D^2+D)(D^3+D) - DD^4 = D^5 + D^4 + D^3 + D^2 - D^5$

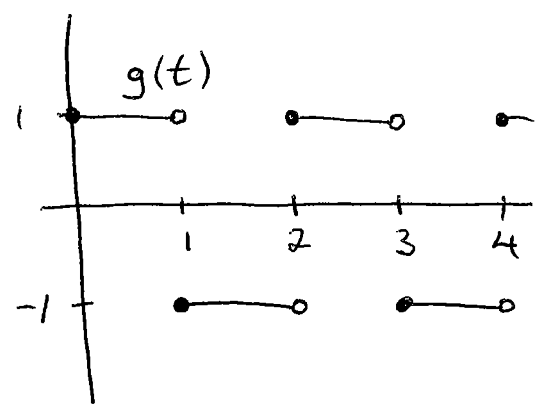
$= \boxed{D^4 + D^3 + D^2}$

3b. (1) How many parameters will a general solution to the system above have?

four (= degree of op. det.)

4. (5) The function $f(t)$ pictured on the left has Laplace transform $\frac{1}{s^2}\tanh(\frac{s}{2}) + \frac{1}{s}$. Find the Laplace transform of $g(t)$, pictured on the right.





$g(t) = f'(t)$. So

$\mathcal{L}\{g\} = \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

$= s\left(\frac{1}{s^2}\tanh\left(\frac{s}{2}\right) + \frac{1}{s}\right) - 1$

$= \boxed{\frac{1}{s}\tanh\left(\frac{s}{2}\right)}$

5. (4,4,4,4) Find the following Laplace transforms and inverse Laplace transforms:

(a) $\mathcal{L}\left\{\frac{\cosh(5t)}{e^{3t}}\right\}$

$$\mathcal{L}\{e^{-3t}\cosh(5t)\} = \boxed{\frac{s+3}{(s+3)^2-25}}$$

because $\mathcal{L}\{\cosh(5t)\} = \frac{s}{s^2-25}$

(b) $\mathcal{L}\{g(t)\}$, where $g'(t) = \frac{\cosh(5t)}{e^{3t}}$

If $g(t) = \int_0^t \frac{\cosh(5\tau)}{e^{3\tau}}\,d\tau$, we have

$$\boxed{\mathcal{L}\{g\} = \frac{1}{s}\frac{(s+3)}{(s+3)^2-25}}$$

Or more generally, $\mathcal{L}\{g'\} = s\mathcal{L}\{g\} - g(0)$

So $\frac{s+3}{(s+3)^2-25} = s\mathcal{L}\{g\} - g(0)$

$$\boxed{\mathcal{L}\{g\} = \frac{1}{s}\frac{(s+3)}{(s+3)^2-25} + \frac{g(0)}{s}}$$

(c) $\mathcal{L}^{-1}\left\{\frac{s+14}{(s-3)^2+16}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+16} + \frac{17}{(s-3)^2+16}\right\}$$

$$= \boxed{e^{3t}\cos(4t) + \frac{17}{4}e^{3t}\sin(4t)}$$

(d) $\mathcal{L}^{-1}\left\{\dfrac{5s^2+2}{(s^2+1)(s)}\right\}$ $\quad \dfrac{5s^2+2}{(s^2+1)(s)} = \dfrac{As+B}{s^2+1} + \dfrac{C}{s}$

$$5s^2+2 = (As+B)s + C(s^2+1)$$

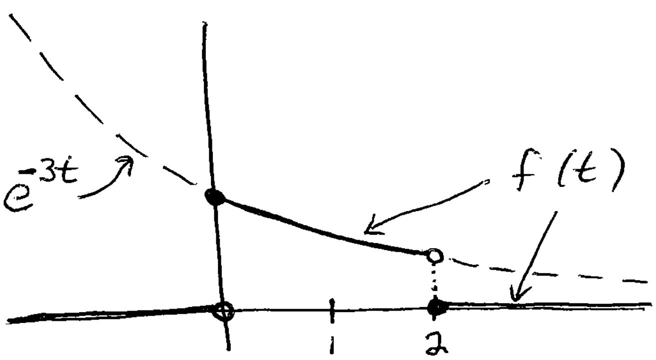
$\underline{s=0}$ $\quad 2 = C$

$\underline{s=i}$ $\quad -3 = -A + Bi, \quad \Rightarrow A = 3, \quad B = 0$

$$\mathcal{L}^{-1}\left\{ \frac{3s}{s^2+1} + \frac{2}{s} \right\}$$

$$= \boxed{3\cos t + 2}$$

6. (5) Using the definition, find the Laplace transform of the function $f(t)$ shown below.



$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st} e^{-3t}\,dt = \int_0^2 e^{-(s+3)t}\,dt$$

$$= \left[\frac{-1}{s+3} e^{-(s+3)t} \right]_0^2$$

$$= \boxed{\frac{-1}{s+3} e^{-2(s+3)} + \frac{1}{s+3}}$$

7. (8) Use the Laplace transform to solve the following initial value problem:

$$x'' + 8x' + 15x = 0, \quad x(0) = 4, \quad x'(0) = -5$$

$\mathcal{L}\{x\} = X(s)$

$\mathcal{L}\{x'\} = s X(s) - 4$

$\mathcal{L}\{x''\} = s^2 X(s) - 4s + 5$

$(s^2 X(s) - 4s + 5) + 8(s X(s) - 4) + 15 X(s) = 0$

$(s^2 + 8s + 15) X(s) = 4s + 27$

$X(s) = \dfrac{4s+27}{s^2+8s+15} = \dfrac{4s+27}{(s+3)(s+5)}$

$\dfrac{4s+27}{(s+3)(s+5)} = \dfrac{A}{s+3} + \dfrac{B}{s+5}$

$4s+27 = A(s+5) + B(s+3)$

$\underline{s=-3}$ $\quad 15 = 2A, \quad A = 15/2$

$\underline{s=-5}$ $\quad 7 = -2B, \quad B = -7/2$

$X(s) = \dfrac{15/2}{s+3} - \dfrac{7/2}{s+5}$

$\boxed{x(t) = \frac{15}{2} e^{-3t} - \frac{7}{2} e^{-5t}}$

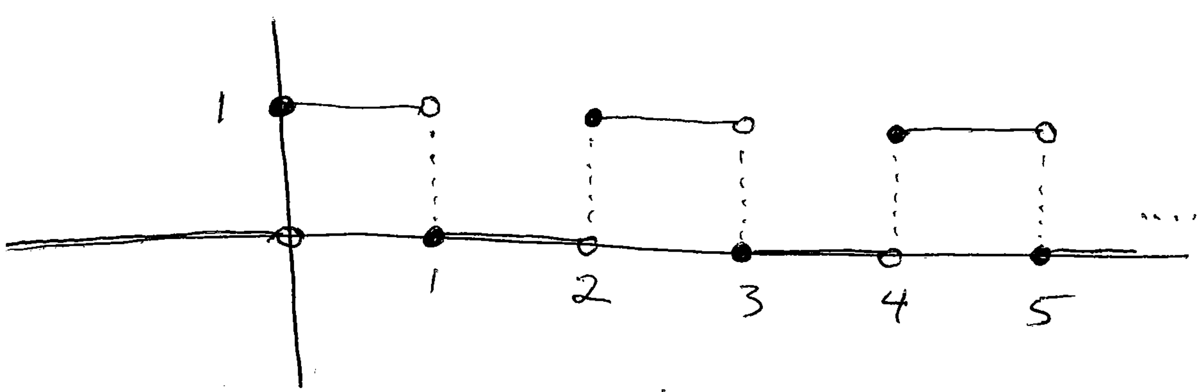
8. (4) Derive the formula for $\mathcal{L}\{f''(t)\}$ by applying the formula for $\mathcal{L}\{f'(t)\}$ twice.

formula $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

Now

$$\mathcal{L}\{f''\} = \mathcal{L}\{(f')'\}$$
$$= s\mathcal{L}\{f'\} - f'(0)$$
$$= s(s\mathcal{L}\{f\} - f(0)) - f'(0)$$
$$= s^2\mathcal{L}\{f\} - sf(0) - f'(0)$$

Bonus. (3) Let $u_a(t)$ be the step function at a. That is, $u_a(t)$ is 0 for $t < a$ and 1 for $t \geq a$. Draw the graph of the function $f(t) = \sum_{n=0}^{\infty}(-1)^n u_n(t)$.



Function	Transform
$f(t)$	$F(s)$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$
$e^{at} f(t)$	$F(s-a)$
t^n	$\frac{n!}{s^{n+1}}$
t^a	$\frac{\Gamma(a+1)}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$