1. (5) Transform the following equation into an equivalent first order system of differential equations:

$$8x' - 12x^{(4)} + 3x'' + x = \cos t$$

$$\boxed{\begin{aligned} &x_1 = x' \\ &x_2 = x_1' \\ &x_3 = x_2' \\ &8x_1 - 12x_3' + 3x_2 + x = \cos t \end{aligned}}$$

2. (7) Use the method of elimination to transform the following system into a single differential equation involving t, x, x', x'', etc. (with no y). Do not solve the equation, but write it in the usual way, without operators.

$$2x' = y + 4x - y' + \cos t$$

$$x + y' = 3y$$

$$\begin{cases} (2D-4)x + (D-1)y = \cos t \\ x + (D-3)y = 0 \end{cases}$$

$$(D-3)(2D-4)x + (D-3)(D-1)y = (D-3)(\cos t)$$

$$-\quad (D-1)x + (D-1)(D-3)y = (D-1)(0)$$

$$(D-3)(2D-4)x - (D-1)x = (D-3)(\cos t) - 0$$

$$(2D^2 - 10D + 12)x - (D-1)x = -\sin t - 3\cos t$$

$$\boxed{2x'' - 11x' + 13x = -\sin t - 3\cos t}$$

3a. (4) Find the operational determinant of the following system:

$$D^2x + Dx + D^4y = 0$$

$$Dx + D^3y + Dy = 0$$

$(D^2+D)x + D^4y = 0$

$Dx + (D^3+D)y = 0$

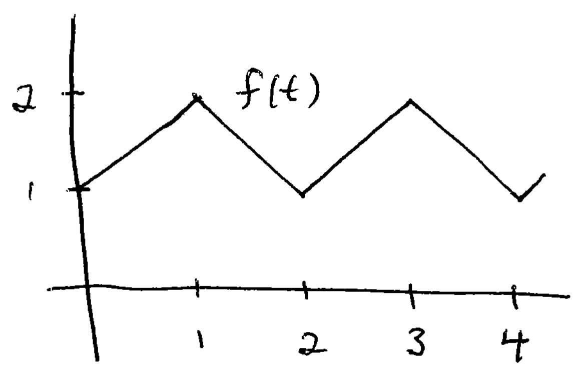
Op. det. $= (D^2+D)(D^3+D) - DD^4 = D^5 + D^4 + D^3 + D^2 - D^5$

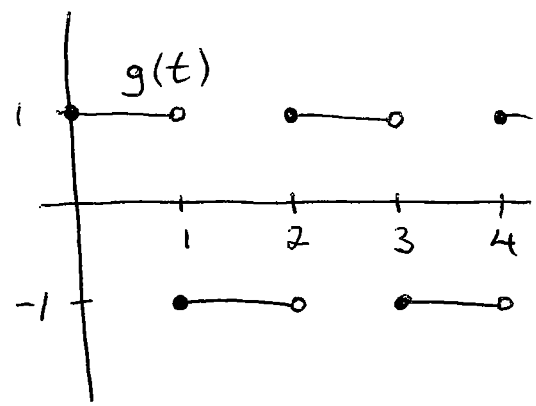
$= \boxed{D^4 + D^3 + D^2}$

3b. (1) How many parameters will a general solution to the system above have?

$\boxed{\text{four}}$ (= degree of op. det.)

4. (5) The function $f(t)$ pictured on the left has Laplace transform $\frac{1}{s^2}\tanh(\frac{s}{2}) + \frac{1}{s}$. Find the Laplace transform of $g(t)$, pictured on the right.





$\underline{g(t) = f'(t)}$. So

$\mathcal{L}\{g\} = \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

$= s\left(\frac{1}{s^2}\tanh(\frac{s}{2}) + \frac{1}{s}\right) - 1$

$= \boxed{\frac{1}{s}\tanh(\frac{s}{2})}$

5. (4,4,4,4) Find the following Laplace transforms and inverse Laplace transforms:

(a) $\mathcal{L}\left\{t^{3/2}e^{-4t}\right\}$

$$\mathcal{L}\{t^{3/2}\} = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{3/2\,\Gamma(3/2)}{s^{5/2}} = \frac{(3/2)(1/2)\sqrt{\pi}}{s^{5/2}}$$

So

$$\boxed{\mathcal{L}\{t^{3/2}e^{-4t}\} = \frac{3/4\sqrt{\pi}}{(s+4)^{5/2}}}$$

(b) $\mathcal{L}\{g(t)\}$, where $g'(t) = t^{3/2}e^{-4t}$

If $g(t) = \int_0^t \tau^{3/2}e^{-4\tau}\,d\tau$, we have

$$\boxed{\mathcal{L}\{g\} = \frac{1}{s}\,\frac{3/4\sqrt{\pi}}{(s+4)^{5/2}}}$$

Or more generally, $\mathcal{L}\{g'\} = s\mathcal{L}\{g\} - g(0)$

So $\frac{3/4\sqrt{\pi}}{(s+4)^{5/2}} = s\mathcal{L}\{g\} - g(0)$

$$\boxed{\mathcal{L}\{g\} = \frac{1}{s}\,\frac{3/4\sqrt{\pi}}{(s+4)^{5/2}} + \frac{g(0)}{s}}$$

(c) $\mathcal{L}^{-1}\left\{\frac{s}{(s-3)^2+4}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{s-3}{(s-3)^2+4} + \frac{3}{(s-3)^2+4}\right\}$$

$$= \boxed{e^{3t}\cos(2t) + \frac{3}{2}e^{3t}\sin(2t)}$$

(d) $\mathcal{L}^{-1}\left\{\dfrac{5s^2+2}{(s^2+1)(s)}\right\}$ $\qquad \dfrac{5s^2+2}{(s^2+1)(s)} = \dfrac{As+B}{s^2+1} + \dfrac{C}{s}$

$$5s^2+2 = (As+B)s + C(s^2+1)$$

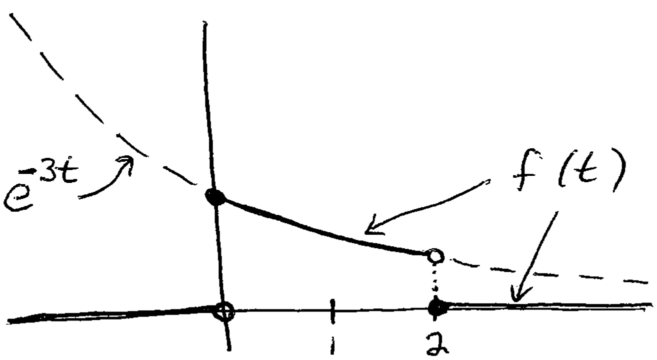
$\underline{s=0}$ $\qquad 2 = C$

$\underline{s=i}$ $\qquad -3 = -A + Bi, \quad \Rightarrow A = 3, \quad B = 0$

$$\mathcal{L}^{-1}\left\{ \frac{3s}{s^2+1} + \frac{2}{s} \right\}$$

$$= \boxed{3\cos t + 2}$$

6. (5) Using the definition, find the Laplace transform of the function $f(t)$ shown below.



$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st} e^{-3t}\,dt = \int_0^2 e^{-(s+3)t}\,dt$$

$$= \left[\frac{-1}{s+3} e^{-(s+3)t} \right]_0^2$$

$$= \boxed{\frac{-1}{s+3} e^{-2(s+3)} + \frac{1}{s+3}}$$

7. (8) Use the Laplace transform to solve the following initial value problem:

$$x'' + 8x' + 15x = 0, \quad x(0) = 2, \quad x'(0) = -3$$

$\mathcal{L}\{x\} = X(s)$

$\mathcal{L}\{x'\} = sX(s) - 2$

$\mathcal{L}\{x''\} = s^2X(s) - 2s + 3$

$$(s^2X(s) - 2s + 3) + 8(sX(s) - 2) + 15X(s) = 0$$

$$(s^2 + 8s + 15)X(s) = 2s + 13$$

$$X(s) = \frac{2s+13}{s^2+8s+15} = \frac{2s+13}{(s+3)(s+5)}.$$

$$\frac{2s+13}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

$$2s + 13 = A(s+5) + B(s+3)$$

$\underline{s=-3}$ $\quad 7 = 2A, \quad A = 7/2$

$\underline{s=-5}$ $\quad 3 = -2B, \quad B = -3/2$

$$X(s) = \frac{7/2}{(s+3)} - \frac{3/2}{(s+5)}$$

$$\boxed{x(t) = \tfrac{7}{2}e^{-3t} - \tfrac{3}{2}e^{-5t}}$$

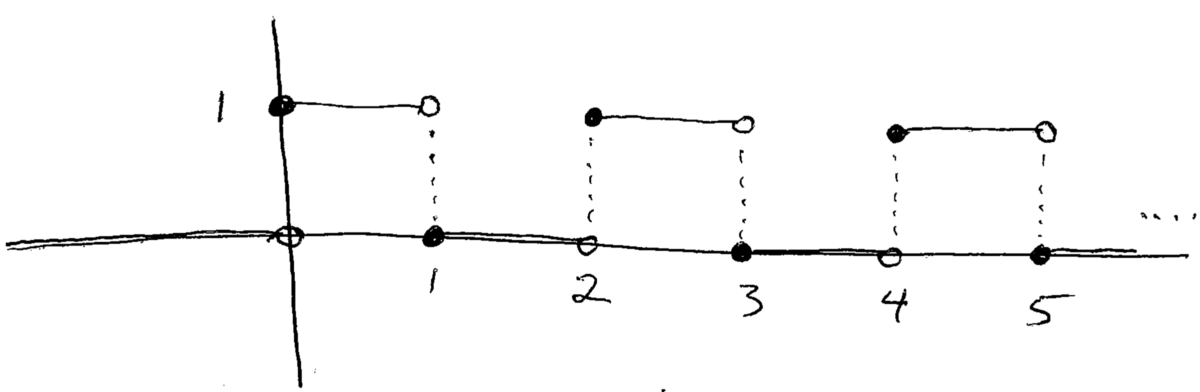
8. (4) Derive the formula for $\mathcal{L}\{f''(t)\}$ by applying the formula for $\mathcal{L}\{f'(t)\}$ twice.

formula $\quad \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$

Now

$$\begin{aligned}
\mathcal{L}\{f''\} &= \mathcal{L}\{(f')'\} \\
&= s\mathcal{L}\{f'\} - f'(0) \\
&= s(s\mathcal{L}\{f\} - f(0)) - f'(0) \\
&= s^2\mathcal{L}\{f\} - sf(0) - f'(0)
\end{aligned}$$

Bonus. (3) Let $u_a(t)$ be the step function at a. That is, $u_a(t)$ is 0 for $t < a$ and 1 for $t \geq a$. Draw the graph of the function $f(t) = \sum_{n=0}^{\infty}(-1)^n u_n(t)$.



Function	Transform
$f(t)$	$F(s)$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$
$e^{at} f(t)$	$F(s-a)$
t^n	$\frac{n!}{s^{n+1}}$
t^a	$\frac{\Gamma(a+1)}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$