1. (3) What does it mean for three functions $y_1(x)$, $y_2(x)$, $y_3(x)$ to be *linearly dependent*?

there are constants A, B, C (not all zero) such that $Ay_1(x) + By_2(x) + Cy_3(x) = 0$.

2. (3,5) Suppose the functions $y_1(x) = x$, $y_2(x) = x^2$, and $y_3(x) = x^3$ are solutions to a linear homogeneous differential equation.

(a) Write down two more solutions to the differential equation.

$$y_4(x) = x + x^2$$

$$y_5(x) = x^2 + x^3$$

(any linear combination of x, x^2, x^3 is a solution)

(b) Use the Wronskian to determine whether y_1, y_2, and y_3 are linearly independent.

$$W(y_1, y_2, y_3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x \begin{vmatrix} 2x & 3x^2 \\ 2 & 6x \end{vmatrix} - x^2 \begin{vmatrix} 1 & 3x^2 \\ 0 & 6x \end{vmatrix} + x^3 \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix}$$

$$= 12x^3 - 6x^3 - 6x^3 + 2x^3$$

$$= 2x^3 \neq 0$$

hence linearly independent.

3. (6) Solve the initial value problem $y'' - 4y' + 3y = 0,\ \ y(0) = 7,\ \ y'(0) = 11.$

Char. equation: $r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0$

$r = 3, 1$

$y = Ae^{3x} + Be^{x} \longrightarrow 7 = Ae^{0} + Be^{0}$

$y' = 3Ae^{3x} + Be^{x} \longrightarrow 11 = 3Ae^{0} + Be^{0}$

$$\begin{aligned} 7 &= A + B \\ 11 &= 3A + B \\ \hline -4 &= -2A \\ A &= 2 \end{aligned} \qquad 7 = 2 + B \qquad B = 5$$

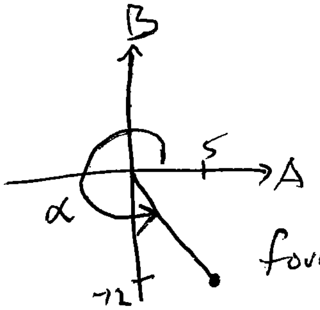
$$\boxed{y(x) = 2e^{3x} + 5e^{x}}$$

4. (6) Convert the function

$$x(t) = \underbrace{5}_{A}\cos(13t) - \underbrace{12}_{B}\sin(13t)$$

into the form $x(t) = C\cos(\omega t - \alpha)$. Use an exact expression (possibly involving $\tan^{-1}$) for α, rather than a decimal value.

$C = \sqrt{A^2 + B^2} = \sqrt{25 + 144} = 13$



fourth quadrant; $\alpha = \tan^{-1}\left(\frac{B}{A}\right) + 2\pi$

$$\boxed{x(t) = 13\cos\left(13t - \left(\tan^{-1}\left(\tfrac{-12}{5}\right) + 2\pi\right)\right)}$$

5. (5,3) This problem concerns the differential equation $y'' + 16y = e^{3x}$.

(a) Using the method of undetermined coefficients, find a particular solution to the equation.

use $y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

$y'' + 16y = e^{3x}$ becomes

$9Ae^{3x} + 16Ae^{3x} = e^{3x}$

$25A = 1$

$A = \frac{1}{25}$

$$\boxed{y = \frac{1}{25}e^{3x}}$$

(b) Find a general solution to the equation.

$r^2 + 16 = 0$; $r = \pm 4i$

$y_c = A\cos(4x) + B\sin(4x)$

$$y = y_c + y_p = \boxed{A\cos(4x) + B\sin(4x) + \frac{1}{25}e^{3x}}$$

6. (4,4) Consider the differential equation

$$y^{(5)} - 6y^{(4)} + 21y^{(3)} - 62y'' + 108y' - 72y = x^2e^{2x} + x\sin(3x).$$

(a) Using the fact that $r^5 - 6r^4 + 21r^3 - 62r^2 + 108r - 72 = (r-2)^3(r^2+9)$, find a general solution to the associated homogeneous equation. (That is, find the complementary solution y_c.)

$r = 2, 2, 2, \pm 3i$

$$\boxed{y_c = Ae^{2x} + Bxe^{2x} + Cx^2e^{2x} + D\cos(3x) + E\sin(3x)}$$

(b) Using the table below, set up the appropriate form of a particular solution (but do not determine the values of the coefficients).

$f(x)$	y_p
$P_m = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^se^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1x + \cdots + A_mx^m)\cos kx + (B_0 + B_1x + \cdots + B_mx^m)\sin kx]$

← use these two

$y_p = x^3(A_0 + A_1x + A_2x^2)e^{2x} + x$[redacted]$(B_0 + B_1x)\cos(3x)$
$+ x$[redacted]$(C_0 + C_1x)\sin(3x)$

7. (5,3) Recall the equation $mx'' + cx' + kx = 0$ for free motion of a mass-spring-dashpot system.

(a) Find a general solution for the position function $x(t)$ when $m = 1$, $c = 6$, $k = 13$. Is this system underdamped or overdamped?

$$x'' + 6x' + 13x = 0$$

$$r^2 + 6r + 13 = 0$$

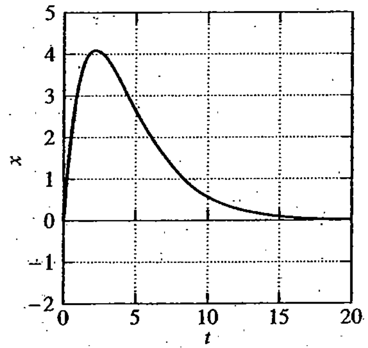
$$r = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2}$$

$$r = -3 \pm 2i$$

$$x(t) = e^{-3t}(A\cos(2t) + B\sin(2t))$$

underdamped

(b) Suppose the damping constant c is changed, and a solution to the new system has the graph shown below. Was c increased or decreased? Explain briefly how you know.



The picture shows a critically damped or overdamped system. Thus c has increased.

8. (8) Consider the endpoint problem

$$y'' + 2y' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

Is $\lambda = 5$ an eigenvalue? If so give an eigenfunction; otherwise say why not.

$y'' + 2y' + 5y = 0$

$r^2 + 2r + 5 = 0 \qquad r = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm 4i}{2}$

$r = -1 \pm 2i$

$y(x) = e^{-x}(A\cos(2x) + B\sin(2x))$.

$\underline{y(0)=0}$: $\quad 0 = A\underbrace{\cos(0)}_{1} + B\underbrace{\sin(0)}_{0}$

$A = 0$

So now $y = e^{-x} B\sin(2x)$.

$\underline{y(\pi)=0}$: $\quad 0 = e^{-\pi} B \underbrace{\sin(2\pi)}_{0}$, true for any B.

So $y = Be^{-x}\sin(2x)$ is a solution to the endpoint problem. $\lambda = 5$ is an eigenvalue.

An eigenfunction is $y = Be^{-x}\sin(2x)$ for any $B \neq 0$.