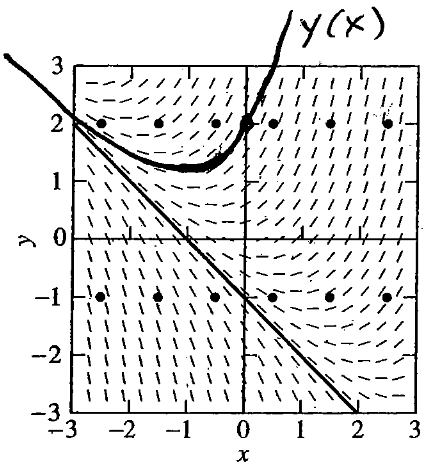
1. (5) The following picture is a slope field for the differential equation $\frac{dy}{dx} = x + y$. Suppose $y(x)$ is a solution to the initial value problem $\frac{dy}{dx} = x + y$, $y(0) = 2$. Without finding y explicitly, what can you say about $y(-250)$? What about $y(250)$?



In the negative x direction, y(x) approaches the line $y = -1 - x$. So $y(-250) \approx 249$.

In the positive x direction, y(x) gets very large, but we can't estimate $y(250)$.

2. (9) This problem concerns the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$.

(a) Use separation of variables to find a general solution to $y' = 3y^{2/3}$.

$$\frac{dy}{dx} = 3y^{2/3}$$

$$\int y^{-2/3}\,dy = \int 3\,dx$$

$$3y^{1/3} = 3x + C$$

$$\boxed{y = (x + \tfrac{1}{3}C)^3}$$

(b) Use your general solution to solve the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$.

$$0 = (2 + \tfrac{1}{3}C)^3$$

$$0 = 2 + \tfrac{1}{3}C$$

$$\tfrac{1}{3}C = -2$$

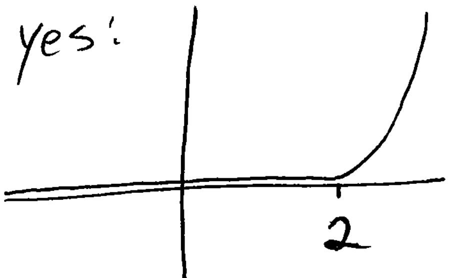
$$\boxed{y = (x - 2)^3}$$

(c) Find another solution to the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$.

The singular solution $\boxed{y = 0}$

(d) Can you find a third solution to the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$?

yes:



$$\boxed{y(x) = \begin{cases} 0 & x \leq 0 \\ (x-2)^3 & x \geq 0 \end{cases}}$$

3. (8) The differential equation $\dfrac{dy}{dx} = (1 - y)\cos x$ is both separable and linear.

(a) Find a general solution using separation of variables.

$$\int \frac{dy}{1-y} = \int \cos x \, dx$$

$$-\ln|1-y| = \sin x + C$$

$$\ln|1-y| = -\sin x - C$$

$$|1-y| = e^{-C} e^{-\sin x}$$

$$1-y = De^{-\sin x}$$

$$\boxed{y = 1 - De^{-\sin x}}$$

(b) Write the equation in linear form, find the integrating factor, and proceed to solve the equation, until you have one quantity equal to the integral of another quantity. (Then stop.)

$$\boxed{\underset{\substack{\uparrow \\ P(x)}}{y' + (\cos x)}y = \underset{\substack{\uparrow \\ Q(x)}}{\cos x}}$$

$$\rho(x) = e^{\int \cos x \, dx} = \underline{e^{\sin x}}$$

$$\frac{d}{dx}\left(e^{\sin x} y\right) = (\cos x) e^{\sin x}$$

$$\boxed{e^{\sin x} y = \int (\cos x) e^{\sin x} \, dx}$$

4. (8) Use an integrating factor to solve the initial value problem $xy' + 3y = 3x^{-3/2}$, $y(1) = 3$.

$$y' + \frac{3}{x}y = 3x^{-5/2}$$

$$\rho(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$$

$$x^3 y' + x^3 \frac{3}{x} y = 3x^3 x^{-5/2}$$

$$\frac{d}{dx}(x^3 y) = 3x^{1/2}$$

$$x^3 y = \int 3x^{1/2}dx = 2x^{3/2} + C$$

$$y = 2x^{-3/2} + Cx^{-3}$$

put in $y(1) = 3$:

$$3 = 2(1) + C(1)$$

$$C = 1$$

$$\boxed{y = 2x^{-3/2} + x^{-3}}$$

5. (5) Use the substitution $p = \frac{dy}{dx}$ to transform the differential equation $y'' = 2y(y')^3$ into a first order differential equation involving only y, p, and $\frac{dp}{dy}$. (Do not solve either equation.)

$$p = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p.$$

$$\boxed{\frac{dp}{dy}p = 2y(p)^3}$$

6. (10) In a certain culture of bacteria, the number of bacteria increased by a factor of five in 10 hours. How long did it take for the population to double?

Note that the population $P(t)$ satisfies the differential equation $\frac{dP}{dt} = kP$. (You may leave your answer as an unsimplified, exact expression.)

$$\frac{dP}{dt} = kP \quad \Rightarrow \quad \int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C$$

$$|P| = e^C e^{kt}$$

$$P(t) = De^{kt}.$$

Note $P(0) = D \cdot e^0 = D$, so D is initial population.

We're told $P(10) = 5D$:

$5D = De^{k(10)}$, so $e^{10k} = 5$,

$10k = \ln 5$, $k = \frac{1}{10}\ln 5$.

Now put in $2D$ for P, find t:

$$2D = De^{(\frac{1}{10}\ln 5)t}$$

$$e^{(\frac{1}{10}\ln 5)t} = 2$$

$$(\tfrac{1}{10}\ln 5)t = \ln 2$$

$$\boxed{t = 10\frac{\ln 2}{\ln 5} \approx 4.31 \text{ hours}}$$

7. (10) Use the substitution $v = y^{-3}$ to find a general solution to $3y + x^3y^4 + 3xy' = 0$.

$3xy' + 3y = -x^3y^4 \rightarrow y' + \frac{1}{x}y = \frac{-x^2}{3}y^4$ Bernoulli, $n=4$.

$v = y^{-3}, \quad y = v^{-1/3}, \quad \frac{dy}{dx} = \frac{-1}{3}v^{-4/3}\frac{dv}{dx}$.

substitute:

$$\frac{-1}{3}v^{-4/3}\frac{dv}{dx} + \frac{1}{x}v^{-1/3} = \frac{-x^2}{3}(v^{-1/3})^4$$

divide by $\frac{-1}{3}v^{-4/3}$:

$$\frac{dv}{dx} - \frac{3}{x}v = x^2$$, linear. $\rho(x) = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = x^{-3}$.

$$\frac{d}{dx}(x^{-3}v) = x^2 \cdot x^{-3}$$

$$x^{-3}v = \int x^{-1}dx = \ln|x| + C.$$

$$v = x^3(\ln|x| + C)$$

$$\boxed{y = [x^3(\ln|x| + C)]^{-1/3} = x^{-1}(\ln|x| + C)^{-1/3}}$$

Bonus. (4) Find a solution to $x^2y'' + y = x^2 + 1$. Trial and error with polynomials. If you put in $y = Ax^2 + Bx + C$ (so $y'' = 2A$) you get

$$x^2(2A) + Ax^2 + Bx + C = x^2 + 1$$

So $3A = 1, \quad B = 0, \quad C = 1$.

$\boxed{y(x) = \frac{1}{3}x^2 + 1}$ is a solution.