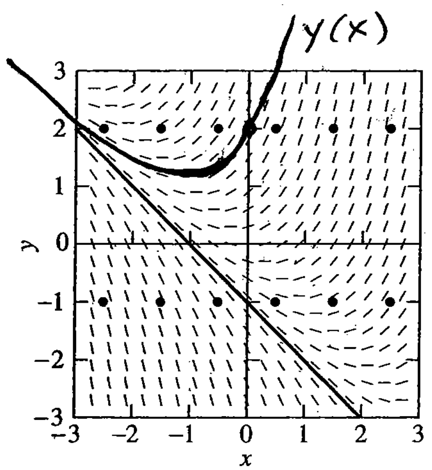
1. (5) The following picture is a slope field for the differential equation $\frac{dy}{dx} = x + y$. Suppose $y(x)$ is a solution to the initial value problem $\frac{dy}{dx} = x + y$, $y(0) = 2$. Without finding y explicitly, what can you say about $y(-250)$? What about $y(250)$?



In the negative x direction, y(x) approaches the line $y = -1 - x$. So $y(-250) \approx 249$.

In the positive x direction, y(x) gets very large, but we can't estimate y(250).

2. (9) This problem concerns the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$.

(a) Use separation of variables to find a general solution to $y' = 3y^{2/3}$.

$$\frac{dy}{dx} = 3y^{2/3}$$

$$\int y^{-2/3}\,dy = \int 3\,dx$$

$$3y^{1/3} = 3x + C$$

$$\boxed{y = (x + \tfrac{1}{3}C)^3}$$

(b) Use your general solution to solve the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$.

$$0 = (2 + \tfrac{1}{3}C)^3$$

$$0 = 2 + \tfrac{1}{3}C$$

$$\tfrac{1}{3}C = -2$$

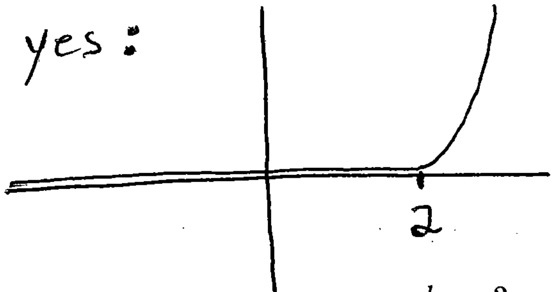
$$\boxed{y = (x-2)^3}$$

(c) Find another solution to the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$.

The singular solution $\boxed{y = 0}$

(d) Can you find a third solution to the initial value problem $y' = 3y^{2/3}$, $y(2) = 0$?

yes:



$$\boxed{y(x) = \begin{cases} 0 & x \le 2 \\ (x-2)^3 & x \ge 2 \end{cases}}$$

3. (8) The differential equation $\dfrac{dy}{dx} = \dfrac{2xy + 2x}{x^2 + 1}$ is both separable and linear.

(a) Write the equation in separable form. That is, separate the variables, and express it as an equation with integrals on both sides (but do not compute these integrals).

$$\frac{dy}{dx} = \frac{2x(y+1)}{x^2+1}$$

$$\boxed{\int \frac{dy}{y+1} = \int \frac{2x\,dx}{x^2+1}}$$

(b) Write the equation in linear form, and find the integrating factor (but do not proceed any further).

$$\frac{dy}{dx} = \frac{2xy}{x^2+1} + \frac{2x}{x^2+1}$$

$$\boxed{\frac{dy}{dx} \underbrace{- \frac{2x}{x^2+1}}_{P(x)} y = \underbrace{\frac{2x}{x^2+1}}_{Q(x)}}$$

$$\rho(x) = e^{\int \frac{-2x}{x^2+1}\,dx}$$

$$\left(\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned} \right)$$

$$\rho(x) = e^{\int -\frac{du}{u}} = e^{-\ln(x^2+1)}$$

$$= \boxed{(x^2+1)^{-1}}$$

4. (8) Use an integrating factor to solve the initial value problem $3xy' + y = 12x$, $y(1) = 4$.

$$y' + \frac{1}{3x}y = 4$$

$$\rho(x) = e^{\int \frac{1}{3x}dx} = e^{\frac{1}{3}\ln x} = x^{1/3}$$

$$x^{1/3}\left(y' + \frac{1}{3x}y\right) = 4x^{1/3}$$

$$\frac{d}{dx}\left(x^{1/3}y\right) = 4x^{1/3}$$

$$x^{\frac{1}{3}}y = \int 4x^{1/3}dx = 3x^{4/3} + C$$

$$y = 3x + Cx^{-1/3}.$$

put in $y(1) = 4$:

$$4 = 3(1) + C(1)$$

$$C = 1$$

$$\boxed{y = 3x + x^{-1/3}}$$

5. (5) Use the substitution $p = \frac{dy}{dx}$ to transform the differential equation $y'' = 2y(y')^3$ into a first order differential equation involving only y, p, and $\frac{dp}{dy}$. (Do not solve either equation.)

$$p = \frac{dy}{dx}, \quad y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = \frac{dp}{dy}p.$$

$$\boxed{\frac{dp}{dy}p = 2y(p)^3}$$

6. (10) A pitcher of buttermilk initially at 25° C is to be cooled by setting it on the front porch, where the temperature is 0° C. Suppose that the temperature of the buttermilk has dropped to 15° C after 20 min. When will it be at 5° C?

According to Newton's Law of cooling, the temperature $T(t)$ of the buttermilk satisfies the differential equation $\frac{dT}{dt} = kT$ in this situation. (You may leave your answer as an unsimplified, exact expression.)

$$\frac{dT}{dt} = kT \Rightarrow \int \frac{dT}{T} = \int k\,dt$$

$$\underline{\ln|T| = kt + C}$$

We have $T(0) = 25$, $T(20) = 15$:

$$\ln|25| = k(0) + C \Rightarrow \underline{C = \ln|25|}$$

$$\ln|15| = k(20) + \ln|25|$$

$$\Rightarrow \underline{k = \frac{1}{20}(\ln 15 - \ln 25)}$$

Now put in $T = 5$, solve for t:

$$\ln|5| = \frac{1}{20}(\ln 15 - \ln 25)t + \ln 25$$

$$\boxed{t = 20\frac{(\ln 5 - \ln 25)}{(\ln 15 - \ln 25)} \approx 63 \text{ minutes}}$$

7. (10) Use the substitution $v = y/x$ to find a general solution to $x^3y' = x^2y - y^3$.

$$y' = \frac{y}{x} - \frac{y^3}{x^3}$$

$$v = \frac{y}{x} \Rightarrow y = xv$$

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$x\frac{dv}{dx} + v = v - v^3$$

$$x\frac{dv}{dx} = -v^3$$

$$\int v^{-3}\,dv = \int \frac{-dx}{x}$$

$$\frac{-1}{2}v^{-2} = -\ln|x| + C$$

$$\frac{1}{v^2} = 2\ln|x| - 2C$$

$$v = (2\ln|x| - 2C)^{-1/2}$$

$$\boxed{y = \frac{x}{\sqrt{2\ln|x| - 2C}}}$$

Bonus. (4) Find a solution to $x^2y'' + y = x^2 + 1$. Trial and error with polynomials. If you put in $y = Ax^2 + Bx + C$, (so $y'' = 2A$) we get

$$x^2(2A) + Ax^2 + Bx + C = x^2 + 1.$$

So $3A = 1$, $B = 0$, $C = 1$.

$\boxed{y(x) = \frac{1}{3}x^2 + 1}$ is a solution.