1a. (5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ by using the convolution formula and calculating the convolution.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\} = 1 * \sin t = \int_0^t 1\cdot \sin(t-\tau)\,d\tau$$

$$u = t-\tau,\ du = -d\tau \Rightarrow \qquad = \int_t^0 -\sin(u)\,du$$

$$= \left[\cos(u)\right]_t^0 = \cos(0) - \cos(t) = \boxed{1-\cos t}$$

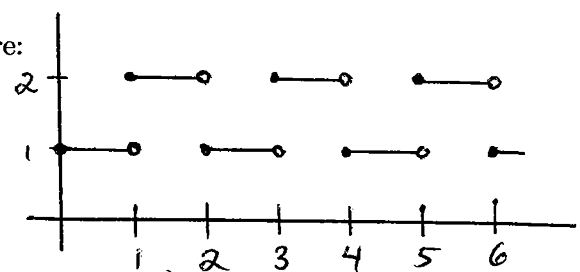
1b. (5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ by using a different formula involving an integral.

Use: $\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$:

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\} = \int_0^t \sin\tau\,d\tau = \left[-\cos\tau\right]_0^t$$

$$= -\cos(t) + \cos(0) = \boxed{1-\cos t}$$

1c. (5) Find the Laplace transform of the function shown here:



It's periodic with period 2.

So $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)\,dt$

$$= \frac{1}{1-e^{-2s}}\left(\int_0^1 e^{-st}\cdot 1\cdot dt + \int_1^2 e^{-st}\cdot 2\cdot dt\right)$$

$$= \frac{1}{1-e^{-2s}}\left(\left[\frac{-1}{s}e^{-st}\right]_0^1 + \left[\frac{-2}{s}e^{-st}\right]_1^2\right)$$

$$= \boxed{\frac{1}{1-e^{-2s}}\left(\frac{-1}{s}e^{-s} + \frac{1}{s} - \frac{2}{s}e^{-2s} + \frac{2}{s}e^{-s}\right)}$$

2. This problem concerns the differential equation $y'' + 4y' + 3y = 2e^{-x}$.

(a) (3) Use the characteristic equation to find a general solution to the associated homogeneous equation.

$r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$r = -1, -3$

$$\boxed{y_c = Ae^{-x} + Be^{-3x}}$$

(b) (5) Use the method of undetermined coefficients to obtain a trial solution, and put it in the equation to determine a particular solution. Refer to the back page to determine your trial solution.

Try $y_p = Axe^{-x}$ (there is duplication)

$y_p' = -Axe^{-x} + Ae^{-x}$

$y_p'' = Axe^{-x} - Ae^{-x} - Ae^{-x}$

plug in: $Axe^{-x} - 2Ae^{-x} - 4Axe^{-x} + 4Ae^{-x} + 3Axe^{-x} = 2e^{-x}$

$2Ae^{-x} = 2e^{-x} \Rightarrow A = 1$ $\boxed{y_p = xe^{-x}}$

(c) (5) Use the solutions found above to write down a general solution of $y'' + 4y' + 3y = 2e^{-x}$, and find the solution that satisfies the initial conditions $y(0) = 0$, $y'(0) = -3$.

$y = y_c + y_p = Ae^{-x} + Be^{-3x} + xe^{-x}$

$y(0) = 0 \Rightarrow 0 = A + B. \Rightarrow B = -A$

$y' = -Ae^{-x} - 3Be^{-3x} + e^{-x} - xe^{-x}$

$y'(0) = -3 \Rightarrow -3 = -A - 3B + 1$

$-4 = -A + 3A = 2A$

$A = -2, B = 2$

$$\boxed{y = -2e^{-x} + 2e^{-3x} + xe^{-x}}$$

3a. (3) Find the operational determinant of the following system:

$$D^2x + Dx + D^2y = 0$$

$$D^2x + D^3y - Dy = 0$$

$$(D^2+D)x + D^2y = 0$$
$$D^2x + (D^3-D)y = 0$$

$$\text{Op. det} = (D^2+D)(D^3-D) - D^2D^2 = D^5 + D^4 - D^3 - D^2 - D^4$$
$$= \boxed{D^5 - D^3 - D^2}$$

3b. (1) How many parameters will a general solution to the system above have?

$\boxed{\text{five}}$ (= degree of op. det.)

4. (8) Use an integrating factor to solve the linear differential equation $xy' - 3y = x^3$.

$$y' \underbrace{- \frac{3}{x}}_{P(x)} y = \underbrace{x^2}_{Q(x)} \qquad \text{Int. factor } \rho(x) = e^{\int \frac{-3}{x}\,dx}$$

$$= e^{-3\ln x} = e^{\ln(x^{-3})} = x^{-3}.$$

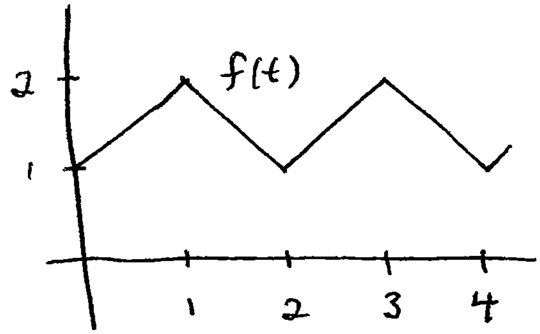
$$x^{-3}y' - x^{-3}\frac{3}{x}y = x^{-3}x^2$$

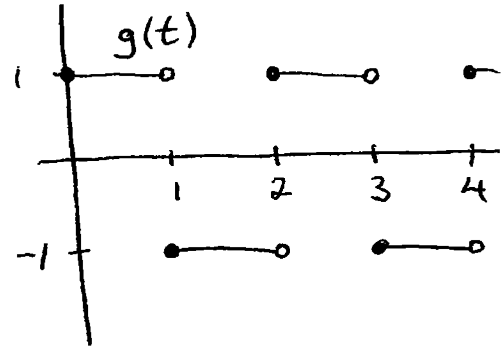
$$\frac{d}{dx}(x^{-3}y) = x^{-1}$$

$$x^{-3}y = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\boxed{y = x^3(\ln|x| + C)}$$

5. (4) The function $g(t)$ pictured on the right has Laplace transform $\frac{1}{s}\tanh(\frac{s}{2})$. Using the derivative rule, find the Laplace transform of $f(t)$, pictured on the left.





$\underline{g = f'}$.

$$\mathcal{L}\{f'\} = s\,\mathcal{L}\{f\} - f(0)$$

$$\frac{1}{s}\tanh\left(\frac{s}{2}\right) = s\,\mathcal{L}\{f\} - 1$$

$$\boxed{\mathcal{L}\{f\} = \frac{1}{s}\left(\frac{1}{s}\tanh\left(\frac{s}{2}\right) + 1\right)}$$

6. (8) Use the substitution $v = y/x$ to solve the differential equation $x^2\frac{dy}{dx} = xy + x^2e^{y/x}$.

$$v = y/x \Rightarrow y = vx, \quad \frac{dy}{dx} = \frac{dv}{dx}x + v\underbrace{\frac{dx}{dx}}_{1}$$

$$x^2\left(\frac{dv}{dx}x + v\right) = x(vx) + x^2e^v$$

$$\frac{dv}{dx}x + v = v + e^v$$

$$\frac{dv}{dx}x = e^v \Rightarrow \int\frac{dv}{e^v} = \int\frac{dx}{x}$$

$$\int e^{-v}dv = \ln|x| + C$$

$$-e^{-v} = \ln|x| + C$$

$$e^{-v} = -\ln|x| - C$$

$$-v = \ln(-\ln|x| - C)$$

$$\boxed{y = -x\ln(-\ln|x| - C)}$$

7. (7) A mass on a spring has position function $x(t)$ satisfying the differential equation $x'' + 4x = f(t)$, where $f(t)$ is the external force. The initial conditions for the motion are $x(0) = 0$ and $x'(0) = 2$. For the external force, the mass is struck by a hammer at time $t = \pi$ with strength 8. Using the Laplace transform, find the position function $x(t)$. Write your answer without step functions (i.e. use cases if necessary) and simplify.

$$x'' + 4x = 8\delta_\pi(t), \quad x(0) = 0, \quad x'(0) = 2$$

Apply $\mathcal{L}$:

$$s^2 X(s) - 2 + 4X(s) = 8e^{-\pi s}$$

$$X(s) = \frac{8e^{-\pi s}}{s^2+4} + \frac{2}{s^2+4}$$

$$x(t) = 4u(t-\pi)\sin(2(t-\pi)) + \sin(2t)$$

$$= \begin{cases} \sin(2t) & t < \pi \\ 4\sin(2t - 2\pi) + \sin(2t) & t \geq \pi \end{cases}$$

$$\boxed{x(t) = \begin{cases} \sin(2t) & t < \pi \\ 5\sin(2t) & t \geq \pi \end{cases}}$$

8. (4,4,4,4) Find the following Laplace and inverse Laplace transforms.

(a) $\mathcal{L}\{3t^4 - t^{3/2}\}$, using the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$$3\frac{4!}{s^5} - \frac{\Gamma(5/2)}{s^{5/2}}$$

$$\Gamma(5/2) = \tfrac{3}{2}\Gamma(3/2) = \tfrac{3}{2}\cdot\tfrac{1}{2}\Gamma(1/2) = \tfrac{3}{4}\sqrt{\pi}$$

$$\boxed{\frac{72}{s^5} - \frac{3\sqrt{\pi}}{4\, s^{5/2}}}$$

(b) $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} \cos t & t < 2\pi \\ 0 & t \geq 2\pi. \end{cases}$ (Hint: write $f(t)$ using a step function.)

$$f(t) = (1 - u(t-2\pi))\cos t$$
$$= \cos t - u(t-2\pi)\cos t$$
$$= \cos t - u(t-2\pi)\cos(t-2\pi)$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{s}{s^2+1} - e^{-2\pi s}\frac{s}{s^2+1}}$$

(c) $\mathcal{L}\{te^{-2t}\sinh t\}$. (Hint: find $\mathcal{L}\{t \sinh t\}$ first.)

$$\mathcal{L}\{\sinh t\} = \frac{1}{s^2-1}$$
$$\mathcal{L}\{t \sinh t\} = \frac{-d}{ds}\left(\frac{1}{s^2-1}\right) = \frac{2s}{(s^2-1)^2}$$

$$\boxed{\mathcal{L}\{te^{-2t}\sinh t\} = \frac{2(s+2)}{((s+2)^2-1)^2}}$$

(d) $\mathcal{L}^{-1}\left\{\frac{9+s}{s^2-1}\right\}$. $\quad \frac{9+s}{s^2-1} = 9\frac{1}{s^2-1} + \frac{s}{s^2-1}$

$$\boxed{9\sinh t + \cosh t}$$

$\underline{f(t)}$	$\underline{F(s)}$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
$e^{at}f(t)$	$F(s-a)$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$f(t)*g(t)$	$F(s)G(s)$ where $f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)d\tau$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$
$f(t)$, period p	$\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt$
t^n	$\frac{n!}{s^{n+1}}$
t^a	$\frac{\Gamma(a+1)}{s^{n+1}}$ where $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$ where $u(t-a)$ is 0 when $t<a$, and 1 when $t\geq a$
$\delta(t-a)$	e^{-as} where $\delta(t-a)$ is a unit impulse at time $t=a$

$\underline{f(x)}$	$\underline{y_p}$
$P_m=b_0+b_1x+b_2x^2+\cdots+b_mx^m$	$x^s(A_0+A_1x+A_2x^2+\cdots+A_mx^m)$
$a\cos kx+b\sin kx$	$x^s(A\cos kx+B\sin kx)$
$e^{rx}(a\cos kx+b\sin kx)$	$x^se^{rx}(A\cos kx+B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0+A_1x+A_2x^2+\cdots+A_mx^m)e^{rx}$
$P_m(x)(a\cos kx+b\sin kx)$	$x^s[(A_0+A_1x+\cdots+A_mx^m)\cos kx$ $+(B_0+B_1x+\cdots+B_mx^m)\sin kx]$