

1a. (5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ by using the convolution formula and calculating the convolution.

$$\mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2+1}\right\} = 1 * \sin t = \int_0^t 1 \cdot \sin(t-\tau) d\tau$$

$$u=t-\tau, du = -d\tau \Rightarrow = \int_t^0 -\sin(u) du$$

$$= [\cos(u)]_t^0 = \cos(0) - \cos(t) = \boxed{1 - \cos t}$$

1b. (5) Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ by using a different formula involving an integral.

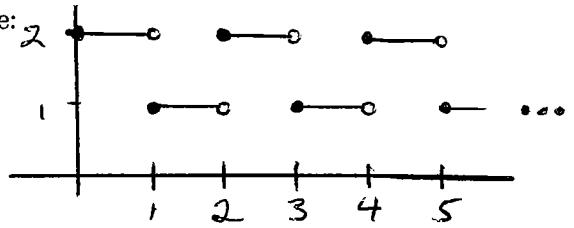
$$\text{use: } \int_0^t f(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s} F(s)\right\} :$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2+1}\right\} = \int_0^t \sin \tau d\tau = [-\cos \tau]_0^t$$

$$= -\cos(t) + \cos(0) = \boxed{1 - \cos t}$$

1c. (5) Find the Laplace transform of the function shown here:

It's periodic with period 2.



$$\text{so } \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2s}} \left(\int_0^1 e^{-st} \cdot 2 dt + \int_1^2 e^{-st} \cdot 1 dt \right)$$

$$= \frac{1}{1-e^{-2s}} \left(\left[\frac{-2}{s} e^{-st} \right]_0^1 + \left[\frac{-1}{s} e^{-st} \right]_1^2 \right)$$

$$= \boxed{\frac{1}{1-e^{-2s}} \left(\frac{-2}{s} e^{-s} + \frac{2}{s} - \frac{1}{s} e^{-2s} + \frac{1}{s} e^{-s} \right)}$$

2. This problem concerns the differential equation $y'' + 4y' + 3y = 2e^{-x}$.

(a) (3) Use the characteristic equation to find a general solution to the associated homogeneous equation.

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0$$

$$r = -1, -3$$

$$y_c = A e^{-x} + B e^{-3x}$$

(b) (5) Use the method of undetermined coefficients to obtain a trial solution, and put it in the equation to determine a particular solution. Refer to the back page to determine your trial solution.

$$\text{Try } y_p = Ax e^{-x} \quad (\text{there is duplication})$$

$$y_p' = -Axe^{-x} + Ae^{-x}$$

$$y_p'' = Axe^{-x} - Ae^{-x} - Ae^{-x}$$

$$\text{plug in: } Axe^{-x} - 2Ae^{-x} - 4Axe^{-x} + 4Ae^{-x} + 3Axe^{-x} = 2e^{-x}$$

$$2Ae^{-x} = 2e^{-x} \Rightarrow A = 1$$

$$y_p = xe^{-x}$$

(c) (5) Use the solutions found above to write down a general solution of $y'' + 4y' + 3y = 2e^{-x}$, and find the solution that satisfies the initial conditions $y(0) = 0$, $y'(0) = -3$.

$$y = y_c + y_p = A e^{-x} + B e^{-3x} + xe^{-x}$$

$$y(0) = 0 \Rightarrow 0 = A + B \Rightarrow B = -A$$

$$y' = -Ae^{-x} - 3Be^{-3x} + e^{-x} - xe^{-x}$$

$$y'(0) = -3 \Rightarrow -3 = -A - 3B + 1$$

$$-4 = -A + 3A = 2A$$

$$A = -2, B = 2$$

$$y = -2e^{-x} + 2e^{-3x} + xe^{-x}$$

3a. (3) Find the operational determinant of the following system:

$$D^2x + x + D^3y = 0$$

$$D^2x + D^3y - Dy = 0$$

$$(D^2 + 1)x + D^3y = 0$$

$$D^2x + (D^3 - D)y = 0$$

$$\begin{aligned} \text{Op. Det.} &= (D^2 + 1)(D^3 - D) - D^2 D^3 \\ &= D^5 + D^3 - D^3 - D - D^5 = \boxed{-D}. \end{aligned}$$

3b. (1) How many parameters will a general solution to the system above have?

one (= degree of op. Det.)

4. (8) Use an integrating factor to solve the linear differential equation $xy' - 3y = x^3$.

$$\begin{aligned} y' - \underbrace{\frac{3}{x}y}_{P(x)} &= \underbrace{x^2}_{Q(x)} & \text{Int. factor } f(x) &= e^{\int \frac{-3}{x} dx} \\ & & &= e^{-3\ln x} = e^{\ln(x^{-3})} = x^{-3}. \end{aligned}$$

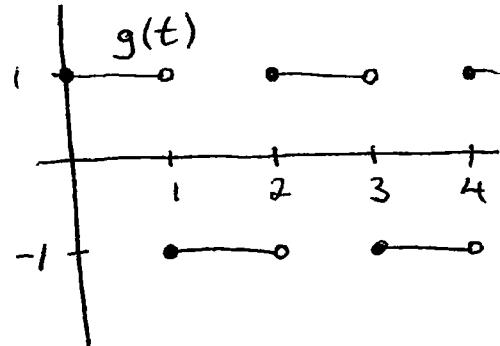
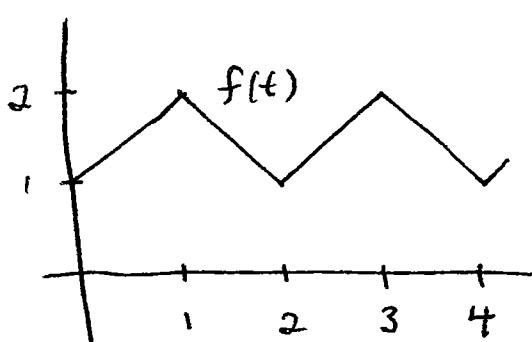
$$x^{-3}y' - x^{-3}\frac{3}{x}y = x^{-3}x^2$$

$$\frac{d}{dx}(x^{-3}y) = x^{-1}$$

$$x^{-3}y = \int \frac{dx}{x} = \ln|x| + C$$

$$y = x^3(\ln|x| + C)$$

5. (4) The function $g(t)$ pictured on the right has Laplace transform $\frac{1}{s} \tanh(\frac{s}{2})$. Using the derivative rule, find the Laplace transform of $f(t)$, pictured on the left.



$$\underline{g = f'}. \quad \mathcal{L}\{f'\} = s \mathcal{L}\{f\} - f(0)$$

$$\frac{1}{s} \tanh\left(\frac{s}{2}\right) = s \mathcal{L}\{f\} - 1$$

$$\boxed{\mathcal{L}\{f\} = \frac{1}{s} \left(\frac{1}{s} \tanh\left(\frac{s}{2}\right) + 1 \right)}$$

6. (8) Use the substitution $v = y/x$ to solve the differential equation $x^2 \frac{dy}{dx} = xy + x^2 e^{y/x}$.

$$v = y/x \Rightarrow y = vx, \quad \frac{dy}{dx} = \frac{dv}{dx}x + v \frac{dx}{dx}$$

$$x^2 \left(\frac{dv}{dx}x + v \right) = x(vx) + x^2 e^v$$

$$\frac{dv}{dx}x + v = v + e^v$$

$$\frac{dv}{dx}x = e^v \Rightarrow \int \frac{dv}{e^v} = \int \frac{dx}{x}$$

$$\int e^{-v} dv = \ln|x| + C \quad \rightarrow \quad -v = \ln(-\ln|x| - C)$$

$$-e^{-v} = \ln|x| + C$$

$$e^{-v} = -\ln|x| - C$$

$$\boxed{y = -x \ln(-\ln|x| - C)}$$

7. (7) A mass on a spring has position function $x(t)$ satisfying the differential equation $x'' + 4x = f(t)$, where $f(t)$ is the external force. The initial conditions for the motion are $x(0) = 0$ and $x'(0) = 2$. For the external force, the mass is struck by a hammer at time $t = \pi$ with strength 4. Using the Laplace transform, find the position function $x(t)$. Write your answer without step functions (i.e. use cases if necessary) and simplify.

$$x'' + 4x = 4\delta_{\pi}(t), \quad x(0) = 0, \quad x'(0) = 2$$

Apply L:

$$s^2 X(s) - 2 + 4X(s) = 4e^{-\pi s}$$

$$X(s) = \frac{4e^{-\pi s}}{s^2 + 4} + \frac{2}{s^2 + 4}$$

$$\begin{aligned} x(t) &= 2u(t-\pi)\sin(2(t-\pi)) + \sin(2t) \\ &= \begin{cases} \sin(2t) & t < \pi \\ 2\sin(2t-2\pi) + \sin(2t) & t > \pi \end{cases} \end{aligned}$$

$$x(t) = \begin{cases} \sin(2t) & t < \pi \\ 3\sin(2t) & t > \pi \end{cases}$$

8. (4,4,4,4) Find the following Laplace and inverse Laplace transforms.

(a) $\mathcal{L}\{4t^3 - 2t^{3/2}\}$, using the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

$$4 \cdot \frac{3!}{s^4} - 2 \cdot \frac{\Gamma(\frac{5}{2})}{s^{\frac{5}{2}}}$$

$$\begin{aligned} \Gamma(\frac{5}{2}) &= \frac{3}{2} \Gamma(\frac{3}{2}) \\ &= \frac{3}{2} \cdot \frac{1}{2} \Gamma(\frac{1}{2}) \\ &= \frac{3}{4} \sqrt{\pi} \end{aligned}$$

$$\boxed{\frac{24}{s^4} - \frac{3\sqrt{\pi}}{2s^{\frac{5}{2}}}}$$

(b) $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} \sin t & t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$. (Hint: write $f(t)$ using a step function.)

$$\begin{aligned} f(t) &= (1 - u(t-2\pi)) \sin t \\ &= \sin t - u(t-2\pi) \sin t \\ &= \sin t - u(t-2\pi) \sin(t-2\pi) \end{aligned}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} - e^{-2\pi s} \frac{1}{s^2+1}$$

(c) $\mathcal{L}\{te^{3t} \sinh t\}$. (Hint: find $\mathcal{L}\{t \sinh t\}$ first.)

$$\begin{aligned} \mathcal{L}\{\sinh t\} &= \frac{1}{s^2-1} \\ \mathcal{L}\{t \sinh t\} &= -\frac{d}{ds} \left(\frac{1}{s^2-1} \right) = \frac{2s}{(s^2-1)^2} \end{aligned}$$

$$\mathcal{L}\{te^{3t} \sinh t\} = \boxed{\frac{2(s-3)}{((s-3)^2-1)^2}}$$

$$(d) \mathcal{L}^{-1}\left\{\frac{9+s}{s^2-4}\right\}. \quad \frac{9+s}{s^2-4} = \frac{9}{2} \cdot \frac{2}{s^2-4} + \frac{5}{s^2-4}$$

$$\boxed{\frac{9}{2} \sinh(2t) + \cosh(2t)}$$

$$\left(= \frac{11}{4}e^{2t} - \frac{7}{4}e^{-2t} \right)$$

<u>$f(t)$</u>	<u>$F(s)$</u>
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
$e^{at}f(t)$	$F(s-a)$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$f(t) * g(t)$	$F(s)G(s)$ where $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$
$f(t)$, period p	$\frac{1}{1-e^{-ps}} \int_0^p e^{-st}f(t)dt$
t^n	$\frac{n!}{s^{n+1}}$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$
e^{at}	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$ where $u(t-a)$ is 0 when $t < a$, and 1 when $t \geq a$
$\delta(t-a)$	e^{-as} where $\delta(t-a)$ is a unit impulse at time $t = a$

<u>$f(x)$</u>	<u>y_p</u>
$P_m = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)$
$a \cos kx + b \sin kx$	$x^s(A \cos kx + B \sin kx)$
$e^{rx}(a \cos kx + b \sin kx)$	$x^s e^{rx}(A \cos kx + B \sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}$
$P_m(x)(a \cos kx + b \sin kx)$	$x^s[(A_0 + A_1x + \cdots + A_mx^m) \cos kx + (B_0 + B_1x + \cdots + B_mx^m) \sin kx]$