**1.** (5 points) Consider the endpoint problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(2\pi) = 0.$$

Is $\lambda = 25$ an eigenvalue? If so give an eigenfunction, and otherwise explain why not.

$y'' + 25y = 0$

$r^2 + 25 = 0$

$r = \pm 5i$ $\qquad y = A\cos(5x) + B\sin(5x)$

$y(0) = 0$:

$0 = A + 0 \implies A = 0$

$y = B\sin(5x)$

$y(2\pi) = 0$:

$0 = B\underbrace{\sin(10\pi)}_{0}$. $\quad$ $B$ is unrestricted.

So $y = B\sin(5x)$ is a solution.

$\boxed{\begin{array}{l}\lambda = 25 \text{ is an eigenvalue} \\ y = B\sin(5x) \text{ is an eigenfunction for any } B \neq 0\end{array}}$

**2.** (5 points) Transform the following system into a first order system:

$$5x^{(3)} - 2x' + y' \sin t = t^2 - y''$$
$$y' - x'' + 14 \sin t = 12x - y.$$

$x_1 = x'$
$x_2 = x_1'$
$y_1 = y'$
$5x_2' - 2x_1 + y_1 \sin t = t^2 - y_1'$
$y_1 - x_2 + 14 \sin t = 12x - y$

**3.** (5 points) Use the method of elimination to transform the following system into a single differential equation involving $t, x, x', x''$, etc (with no $y$). Do not solve the equation, but write it in the usual way, without operators.

$$4x' - 11y = x + 2y' - t^2$$
$$5y - 3x' + 2y' = \cos t$$

$(2D+5)\left[ (4D-1)x + (-2D-11)y = -t^2 \right]$
$+ \; (2D+11)\left[ (-3D)x + (2D+5)y = \cos t \right]$

$(2D+5)(4D-1)x + (2D+11)(-3D)x = (2D+5)(-t^2) + (2D+11)(\cos t)$

$(8D^2 + 18D - 5)x + (-6D^2 - 33D)x = -4t - 5t^2 - 2\sin t + 11\cos t$

$$\boxed{2x''(t) - 15x'(t) - 5x(t) = -4t - 5t^2 - 2\sin t + 11\cos t}$$

**4.** (8 points) Use the method of elimination to solve the following initial value problem:

$$x' = -3x + 2y, \quad y' = -3x + 4y, \quad x(0) = 0, \quad y(0) = 2$$

$①: \; 3\,[\,(D+3)x - 2y = 0\,]$

$②: \; (D+3)\,[\,3x + (D-4)y = 0\,]$

$②-①: \; (D+3)(D-4)y + 6y = 0$

$y'' - y' - 6y = 0 \qquad r^2 - r - 6 = 0$

$(r-3)(r+2) = 0$

$r = 3, -2$

$y = Ae^{3t} + Be^{-2t}$

$y(0) = 2: \qquad 2 = A + B \implies \underline{y = Ae^{3t} + (2-A)e^{-2t}}$

and $y' = 3Ae^{3t} - 2(2-A)e^{-2t}$.

Using orig. equation:

$3x = 4y - y' = 4Ae^{3t} + 4(2-A)e^{-2t} - 3Ae^{3t} + 2(2-A)e^{-2t}$

$= Ae^{3t} + 6(2-A)e^{-2t}$

$\underline{x = \frac{1}{3}Ae^{3t} + 2(2-A)e^{-2t}}$.

$x(0) = 0: \qquad 0 = \frac{1}{3}A + 2(2-A) \implies \frac{5}{3}A = 4$

$\implies A = \frac{12}{5}$.

$$\boxed{\begin{aligned} x(t) &= \tfrac{4}{5}e^{3t} - \tfrac{4}{5}e^{-2t} \\ y(t) &= \tfrac{12}{5}e^{3t} - \tfrac{2}{5}e^{-2t} \end{aligned}}$$

**5.** (12 points) Find the following Laplace transforms and inverse Laplace transforms:

**(a)** $\mathcal{L}\{e^{(1/2)t}\sin 3t\}$

$$\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9}$$

so $$\mathcal{L}\{e^{\frac{1}{2}t}\sin(3t)\} = \boxed{\frac{3}{(s-\frac{1}{2})^2+9}}.$$

**(b)** $\mathcal{L}\{g(t)\}$, where $g'(t) = e^{(1/2)t}\sin(3t)$

$$\frac{3}{(s-\frac{1}{2})^2+9} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0)$$

$$\mathcal{L}\{g(t)\} = \boxed{\frac{1}{s}\left[\frac{3}{(s-\frac{1}{2})^2+9} + g(0)\right]}.$$

**(c)** $\mathcal{L}^{-1}\{\frac{s+1}{(s-2)^2+4}\}$

$$\frac{s+1}{(s-2)^2+4} = \frac{s-2}{(s-2)^2+4} + \frac{3}{(s-2)^2+4}$$

$$\boxed{\cos(2t)e^{2t} + \frac{3}{2}\sin(2t)e^{2t}}$$

**(d)** $\mathcal{L}^{-1}\{\frac{3s+2}{(s^2+4)(s-1)}\}$

$$\frac{3s+2}{(s^2+4)(s-1)} = \frac{As+B}{s^2+4} + \frac{C}{s-1}, \qquad 3s+2 = (As+B)(s-1) + (s^2+4)C$$

$\underline{s=1}$ $\quad 5 = 5C \Rightarrow C = 1.$

$\underline{s=2i}$ $\quad 6i+2 = (2Ai+B)(2i-1) + 0 = -4A + 2Bi - 2Ai - B$

$$6i+2 = (2B-2A)i + (-4A-B)$$

$$6 = 2B-2A, \quad 2 = -4A-B \Rightarrow B = -4A-2$$

$$6 = 2(-4A-2) - 2A \Rightarrow A = -1, \quad B = 2.$$

$$\mathcal{L}^{-1}\{F(s)\} = -\cos(2t) + \sin(2t) + e^t$$

**6.** (8 points) Use the Laplace transform to solve the following initial value problem:

$$x'' - x' - 2x = 0, \quad x(0) = 0, \quad x'(0) = 2$$

Let $\mathcal{L}\{x\} = X(s)$. Then $\mathcal{L}\{x''\} = s^2X(s) - s(0) - 2$

and $\mathcal{L}\{x'\} = sX(s)$

So

$$(s^2X(s) - 2) - (sX(s)) - 2X(s) = 0$$

$$X(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s-2)(s+1)}.$$

$$\frac{2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$2 = A(s+1) + B(s-2)$$

$\underline{s=-1}$ $\quad 2 = -3B, \quad B = \frac{-2}{3}$

$\underline{s=2}$ $\quad 2 = 3A, \quad A = \frac{2}{3}$

$$X(s) = \frac{2}{3}\frac{1}{s-2} + \frac{-2}{3}\frac{1}{s+1}$$

$$\boxed{x(t) = \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t}}$$

**7.** (6 points) Using the definition, find the Laplace transform of $f(t)$, where $f(t) = 0$ for $0 \le t < 2$ and $f(t) = 2$ for $t \ge 2$.

$$\mathcal{L}\{f(t)\} = \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_2^\infty e^{-st} \cdot 2\,dt = 2\left[\frac{-1}{s} e^{-st}\right]_2^\infty$$

$$= \lim_{b\to\infty} \Big( \underbrace{\frac{-2}{s} e^{-sb}}_{\to 0} + \frac{2}{s} e^{-2s} \Big)$$

$$= \boxed{\frac{2}{s} e^{-2s}}$$

**8.** (6 points) Use partial fractions to write the following function as a sum simpler functions, and calculate the coefficients:

$$\frac{s^2+4}{(s-1)^2(s^2+9)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+9}$$

$(*)$ $s^2+4 = A(s-1)(s^2+9) + B(s^2+9) + (Cs+D)(s-1)^2$

$s=1$: $5 = 10B \Rightarrow B = \frac{1}{2}$

$s=3i$: $-9+4 = (3Ci+D)(3i-1)^2 = (3Ci+D)(-9-6i+1)$

$-5 = (3Ci+D)(-6i-8) = 18C - 6Di - 24Ci - 8D$

$= (-24C-6D)i + (18C-8D)$

$0 = -24C - 6D \Rightarrow D = -4C$

$-5 = 18C - 8D = 18C + 32C = 50C \Rightarrow C = \frac{-1}{10}$

$\Rightarrow D = \frac{4}{10}$

diff $(*)$: $2s = A(s-1)(2s) + A(s^2+9) + 2Bs$

$+ (Cs+D)2(s-1) + C(s-1)^2$

$s=1$: $2 = 10A + 2B = 10A + 1 \Rightarrow A = \frac{1}{10}$

| Function | Laplace Transform |
|---|---|
| $f(t)$ | $F(s)$ |
| $af(t)+bg(t)$ | $aF(s)+bG(s)$ |
| $f'(t)$ | $sF(s)-f(0)$ |
| $f^{(n)}(t)$ | $s^nF(s)-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$ |
| $\int_0^t f(\tau)d\tau$ | $\frac{F(s)}{s}$ |
| $e^{at}f(t)$ | $F(s-a)$ |
| $u(t-a)f(t-a)$ | $e^{-as}F(s)$ |
| $(f*g)(t)$ | $F(s)G(s)$ where $(f*g)(t)=\int_0^t f(\tau)g(t-\tau)d\tau$ |
| $tf(t)$ | $-F'(s)$ |
| $t^nf(t)$ | $(-1)^nF^{(n)}(s)$ |
| $\frac{f(t)}{t}$ | $\int_s^\infty F(\sigma)d\sigma$ |
| $f(t)$, period $p$ | $\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $t^a$ | $\frac{\Gamma(a+1)}{s^{n+1}}$ where $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\cos(kt)$ | $\frac{s}{s^2+k^2}$ |
| $\sin(kt)$ | $\frac{k}{s^2+k^2}$ |
| $\cosh(kt)$ | $\frac{s}{s^2-k^2}$ |
| $\sinh(kt)$ | $\frac{k}{s^2-k^2}$ |
| $u(t-a)$ | $\frac{e^{-as}}{s}$ where $u(t-a)$ is 0 when $t<a$, and 1 when $t\geq a$ |
| $\delta(t-a)$ | $e^{-as}$ where $\delta(t-a)$ is a unit impulse at time $t=a$ |