

1. (8 points) Consider the endpoint problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(2) = 0.$$

Is  $\lambda = 5$  an eigenvalue? If so give an eigenfunction, and otherwise explain why not.

$$y'' + 5y = 0 \quad r^2 + 5 = 0 \quad r = \pm \sqrt{5} i$$

$$y = A \cos(\sqrt{5}x) + B \sin(\sqrt{5}x)$$

$$y(0) = 0 : \quad 0 = A + 0 \quad \Rightarrow A = 0$$

$$y = B \sin(\sqrt{5}x).$$

$$y(2) = 0 :$$

$$0 = B \underbrace{\sin(2\sqrt{5})}_{\neq 0}$$

$$\text{so } B = 0$$

Hence the only solution is  $y = 0$ ,

so  $\lambda = 5$  is not an eigenvalue.

2. (8 points) Use the method of elimination to solve the following initial value problem:

$$x' = -3x + 2y, \quad y' = -3x + 4y, \quad x(0) = 0, \quad y(0) = 2$$

$$\textcircled{1}: \quad 3 \left[ (D+3)x - 2y = 0 \right]$$

$$\textcircled{2}: (D+3) \left[ 3x + (D-4)y = 0 \right]$$

$$\textcircled{2} - \textcircled{1}: \quad (D+3)(D-4)y + 6y = 0$$

$$y'' - y' - 6y = 0 \quad r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0$$

$$r = 3, -2$$

$$y = Ae^{3t} + Be^{-2t}$$

$$y(0) = 2 : \quad 2 = A + B \Rightarrow y = Ae^{3t} + (2-A)e^{-2t}$$

$$\text{and } y' = 3Ae^{3t} - 2(2-A)e^{-2t}$$

Using orig. equation:

$$3x = 4y - y' = 4Ae^{3t} + 4(2-A)e^{-2t} - 3Ae^{3t} + 2(2-A)e^{-2t}$$

$$= Ae^{3t} + 6(2-A)e^{-2t}$$

$$x = \frac{1}{3}Ae^{3t} + 2(2-A)e^{-2t}$$

$$x(0) = 0 : \quad 0 = \frac{1}{3}A + 2(2-A) \Rightarrow \frac{5}{3}A = 4$$

$$\Rightarrow A = \frac{12}{5}$$

$$x(t) = \frac{4}{5}e^{3t} - \frac{4}{5}e^{-2t}$$

$$y(t) = \frac{12}{5}e^{3t} - \frac{2}{5}e^{-2t}$$

3. (12 points) Find the following Laplace transforms and inverse Laplace transforms:

(a)  $\mathcal{L}\left\{\frac{1}{t} e^{\frac{1}{2}t} \sin(3t)\right\}$

$$\mathcal{L}\{\sin(3t)\} = \frac{3}{s^2+9}, \text{ so } \mathcal{L}\left\{e^{\frac{1}{2}t} \sin(3t)\right\} = \boxed{\frac{3}{(s-\frac{1}{2})^2+9}}$$

(b)  $\mathcal{L}\{g(t)\}$ , where  $g'(t) = \frac{1}{t} \sin(\frac{1}{2}t) e^{\frac{1}{2}t} \sin(3t)$

$$\frac{s}{(s-\frac{1}{2})^2+9} = \mathcal{L}\{g'\} = s \mathcal{L}\{g\} - g(0)$$

$$\mathcal{L}\{g\} = \boxed{\frac{1}{s} \left[ \frac{s}{(s-\frac{1}{2})^2+9} + g(0) \right]}$$

(c)  $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+4}\right\}$

$$\frac{s}{(s-2)^2+4} = \frac{s-2}{(s-2)^2+4} + \frac{2}{(s-2)^2+4}$$

$$\boxed{e^{2t} \cos(2t) + e^{2t} \sin(2t)}$$

(d)  $\mathcal{L}^{-1}\left\{\frac{3s+2}{(s^2+4)(s-1)}\right\}$

$$\frac{3s+2}{(s^2+4)(s-1)} = \frac{As+B}{s^2+4} + \frac{C}{s-1}, \quad 3s+2 = (As+B)(s-1) + (s^2+4)C$$

$$\underline{s=1}: \quad 5 = 5C \Rightarrow C = 1.$$

$$\underline{s=2i}: \quad 6i+2 = (2Ai+B)(2i-1) + 0 = -4A + 2Bi - 2Ai - B$$

$$6i+2 = (2B-2A)i + (-4A-B)$$

$$6 = 2B-2A, \quad 2 = -4A-B \Rightarrow B = -4A-2$$

$$6 = 2(-4A-2) - 2A \Rightarrow A = -1, B = 2$$

$$\mathcal{L}^{-1}\{F(s)\} = -\cos(2t) + \sin(2t) + e^t$$

- 4 (8 points) Use the Laplace transform to solve the following initial value problem:  
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$$x'' - x' - 2x = 0, \quad x(0) = 0, \quad x'(0) = 2$$

Let  $\mathcal{L}\{x\} = X(s)$ . Then  $\mathcal{L}\{x''\} = s^2 X(s) - s(0) - 2$   
 and  $\mathcal{L}\{x'\} = sX(s)$   
 So

$$(s^2 X(s) - 2) - (sX(s)) - 2X(s) = 0$$

$$X(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s-2)(s+1)}.$$

$$\frac{2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$2 = A(s+1) + B(s-2)$$

$$\underline{s=-1} \quad 2 = -3B, \quad B = -\frac{2}{3}$$

$$\underline{s=2} \quad 2 = 3A, \quad A = \frac{2}{3}$$

$$X(s) = \frac{\frac{2}{3}}{s-2} + \frac{-\frac{2}{3}}{s+1}$$

$$x(t) = \frac{2}{3} e^{2t} - \frac{2}{3} e^{-t}$$

5. (8 points) Using the definition, find the Laplace transform of  $f(t)$ , where  $f(t) = 0$  for  $0 \leq t \leq 2$  and  $f(t) = 2$  for  $t \geq 2$ .

$$\begin{aligned}
 \mathcal{L}\{f(t)\} &= \int_0^2 e^{-st} f(t) dt + \int_2^\infty e^{-st} f(t) dt \\
 &= \int_2^\infty e^{-st} 2 dt = 2 \left[ -\frac{1}{s} e^{-st} \right]_2^\infty \\
 &= \lim_{b \rightarrow \infty} \left( \underbrace{\frac{-2}{s} e^{-sb}}_{\rightarrow 0} + \frac{2}{s} e^{-2s} \right) \\
 &= \boxed{\frac{2}{s} e^{-2s}}
 \end{aligned}$$

6. (7 points) Transform the following system into a first order system:

$$5x^{(3)} - 2x' + y \sin t = t^2 - y''$$

$$y' - x'' + 14e^t = 12x - y.$$

$$x_1 = x'$$

$$x_2 = x_1'$$

$$y_1 = y'$$

$$5x_2' - 2x_1 + y \sin t = t^2 - y_1'$$

$$y_1' - x_2 + 14e^t = 12x - y$$

FUNCTION	LAPLACE TRANSFORM
$f(t)$	$F(s)$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
$e^{at}f(t)$	$F(s-a)$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$(f * g)(t)$	$F(s)G(s)$ where $(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$
$f(t)$ , period $p$	$\frac{1}{1-e^{-ps}} \int_0^p e^{-st}f(t)dt$
$t^n$	$\frac{n!}{s^{n+1}}$
$t^a$	$\frac{\Gamma(a+1)}{s^{a+1}}$ where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$
$e^{at}$	$\frac{1}{s-a}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$ where $u(t-a)$ is 0 when $t < a$ , and 1 when $t \geq a$
$\delta(t-a)$	$e^{-as}$ where $\delta(t-a)$ is a unit impulse at time $t = a$