Name: ID #:

Exam II

Math 3113-005

October 26, 2004

Problem 1:	Problem 5:
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1a. (3) What does it mean for three functions $f_1(x)$, $f_2(x)$, $f_3(x)$ to be *linearly dependent*?

There exist constants A, B, C (not all zero) such that

$$\boxed{A f_1(x) + B f_2(x) + C f_3(x) = 0.}$$

1b. (3) Are the functions $f_1(x) = x - 1$, $f_2(x) = x^2 - x$, $f_3(x) = 1 - x^2$ linearly independent? Explain why or why not.

No, because

$$\boxed{1 \cdot (x-1) + 1 \cdot (x^2 - x) + 1 \cdot (1 - x^2) = 0}.$$

2. (5) Compute the Wronskian of $f_1(x) = e^{-x}$, $f_2(x) = 2x$, and $f_3(x) = 5$. What can you conclude about linear independence?

$$W = \begin{vmatrix} e^{-x} & 2x & 5 \\ -e^{-x} & 2 & 0 \\ e^{-x} & 0 & 0 \end{vmatrix} =$$

$$e^{-x} \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} - 2x \begin{vmatrix} -e^{-x} & 0 \\ e^{-x} & 0 \end{vmatrix} + 5 \begin{vmatrix} -e^{-x} & 2 \\ e^{-x} & 0 \end{vmatrix}$$

$$= 5(-2e^{-x}) = \boxed{-10e^{-x}} \neq 0.$$

So f_1, f_2, f_3 are linearly independent.

3. (4,3) A homogeneous linear differential equation with constant coefficients has characteristic polynomial $r^2(r^2-2r+5)^2(r-5)(r^2-4)$.

(a) Write down a general solution to this differential equation.

roots: $0, 0, \dfrac{2 \pm \sqrt{4-20}}{2}$ (twice), $5, 2, -2$

$\quad\quad = 1 \pm 2i$

$$\boxed{y = A + Bx + Ce^{x}\cos(2x) + De^{x}\sin(2x) + Exe^{x}\cos(2x) + Fxe^{x}\sin(2x) + Ge^{5x} + He^{2x} + Ie^{-2x}}$$

(b) Give an example of initial conditions for this equation which describe exactly one solution.

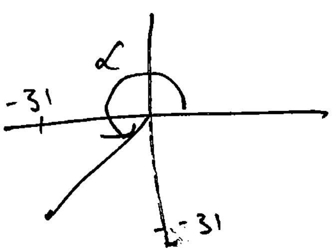
$y(2) = 4$	$y^{(3)}(2) = 0$	$y^{(6)}(2) = 4$
$y'(2) = 3$	$y^{(4)}(2) = 3$	$y^{(7)}(2) = 0$
$y''(2) = 5$	$y^{(5)}(2) = 1$	$y^{(8)}(2) = 2$

linear 9th order equation

4. (5) Convert the function $x(t) = -31\cos(7t) - 31\sin(7t)$ into the form $x(t) = C\cos(\omega t - \alpha)$. Use an exact expression for α, rather than a decimal value.

$A = -31, \quad B = -31$

$C = \sqrt{A^2 + B^2} = \sqrt{31^2 + 31^2} = 31\sqrt{2}$



$\alpha = \tan^{-1}\left(\dfrac{-31}{-31}\right) + \pi = \dfrac{\pi}{4} + \pi$

$= \dfrac{5\pi}{4}$

$$\boxed{x(t) = 31\sqrt{2}\cos\left(7t - \frac{5\pi}{4}\right)}$$

5. (3,4) This problem concerns the differential equation $y''' - 7y'' + 20y' - 24y = 0$.

(a) Verify that $y_1 = e^{3x}$ is a solution.

$y_1' = 3e^{3x}$

$y_1'' = 9e^{3x}$

$y_1''' = 27e^{3x}$

$$\boxed{\begin{aligned} &27e^{3x} - 7(9e^{3x}) + 20(3e^{3x}) - 24(e^{3x}) \\ &= (27 - 63 + 60 - 24)e^{3x} \\ &= 0 \quad \checkmark \end{aligned}}$$

(b) Find a general solution to the equation. Part (a) provides a hint.

$r^3 - 7r^2 + 20r - 24 = 0$

$r = 3$ works, so divide by $r-3$:

$$\begin{array}{r|l} & r^2 - 4r + 8 \\ \hline r-3 & r^3 - 7r^2 + 20r - 24 \\ & r^3 - 3r^2 \\ \hline & \quad -4r^2 + 20r \\ & \quad -4r^2 + 12r \\ \hline & \qquad 8r - 24 \\ & \qquad 8r - 24 \\ \hline & \qquad\quad 0 \end{array}$$

$(r-3)(r^2 - 4r + 8) = 0$

$r = 3, \quad r = \dfrac{4 \pm \sqrt{16 - 32}}{2} = 2 \pm 2i$

$$\boxed{y = Ae^{3x} + Be^{2x}\cos(2x) + Ce^{2x}\sin(2x)}$$

6. (3,3,3,3) Using the table below, what function would you put into the differential equation to find a particular solution? Do not solve for any coefficients.

$f(x)$	y_p
$P_m = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^se^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1x + \cdots + A_mx^m)\cos kx$ $+ (B_0 + B_1x + \cdots + B_mx^m)\sin kx]$

(a) $y'' + 16y = 2e^{4x}$

$r^2 + 16 = 0$

$r = \pm 4i$

$y_c = A\cos(4x) + B\sin(4x)$

try

$\boxed{y_p = Ae^{4x}}$

no duplication

(b) $y'' + 16y = 4\cos(2x) + 3x\sin(5x)$

$y_c = A\cos(4x) + B\sin(4x)$

no duplication

try

$$\boxed{y_p = A\cos(2x) + B\sin(2x) + (C_0 + C_1x)\cos(5x) + (D_0 + D_1x)\sin(5x)}$$

(c) $y'' + 6y' + 9y = 12e^{-3x}$

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$

$r = -3, -3$

$y_c = Ae^{-3x} + Bxe^{-3x}$

try

$\boxed{y_p = x^2Ae^{-3x}}$

duplication

(d) $y'' - 25y = (4 + 2x)e^{5x}$

$r^2 - 25 = 0$

$(r+5)(r-5) = 0$

$r = \pm 5$

$y_c = Ae^{5x} + Be^{-5x}$

try

$\boxed{y_p = x(A_0 + A_1x)e^{5x}}$

duplication

7. (3,3) Recall the equation $mx'' + cx' + kx = 0$ for the motion of a mass on a spring with damping.

(a) Find the position function $x(t)$ when $m = 1$, $c = 2$, and $k = 2$. Is the system overdamped, critically damped, or underdamped?

$$x'' + 2x' + 2 = 0$$

$$r^2 + 2r + 2 = 0$$

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\boxed{x(t) = Ae^{-t}\cos t + Be^{-t}\sin t, \quad \text{underdamped}}$$

(b) You pull the mass to the right 2 units, pause, and let it go. Find the specific position function in this situation.

Initial conditions: $x(0) = 2$, $x'(0) = 0$.

$$2 = Ae^0\cos(0) + Be^0\sin 0$$

$$\underline{2 = A}. \Rightarrow x(t) = 2e^{-t}\cos t + Be^{-t}\sin t$$

$$x'(t) = -2e^{-t}\cos t - 2e^{-t}\sin t - Be^{-t}\sin t + Be^{-t}\cos t$$

$$0 = -2e^0\cos 0 - 2e^0\sin 0 - Be^0\sin 0 + Be^0\cos 0$$

$$0 = -2 + B$$

$$\underline{B = 2}$$

$$\boxed{x(t) = 2e^{-t}\cos t + 2e^{-t}\sin t}$$

8. (4,3) This problem concerns the differential equation $y'' + 9y = 6e^{2x}$.

(a) Using the method of undetermined coefficients, find a particular solution to the equation.

Try $y_p = Ae^{2x}$.

$y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

$4Ae^{2x} + 9(Ae^{2x}) = 6e^{2x}$

$13A = 6$

$A = \frac{6}{13}$

$$\boxed{y_p = \frac{6}{13}e^{2x}}$$

(b) Find a general solution to the equation.

$r^2 + 9 = 0$, $r = \pm 3i$

$y_c = A\cos(3x) + B\sin(3x)$

$$\boxed{y = y_c + y_p = A\cos(3x) + B\sin(3x) + \frac{6}{13}e^{2x}}$$

Extra Credit (3) Use Euler's formula $e^{ix} = \cos x + i \sin x$ to derive a formula for $\cos(\alpha + \beta)$.

$e^{i(\alpha+\beta)} = \underline{\cos(\alpha+\beta)} + i\sin(\alpha+\beta)$

$\parallel$

$e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= \underline{\cos\alpha\cos\beta} + i\sin\alpha\cos\beta + i\cos\alpha\sin\beta + \underline{(-1)\sin\alpha\sin\beta}.$

take real parts!

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$