

Name:

ID #:

Exam II
Math 3113-002
October 25, 2004

Problem 1:

Problem 5:

Problem 2:

Problem 6:

Problem 3:

Problem 7:

Problem 4:

1a. (4) What does it mean for three functions $f_1(x)$, $f_2(x)$, $f_3(x)$ to be *linearly dependent*?

There are constants c_1, c_2, c_3 , not all zero, so that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0.$$

1b. (4) Are the functions $f_1(x) = x - 1$, $f_2(x) = x^2 - x$, $f_3(x) = 1 - x^2$ linearly independent? Explain why or why not.

No, because

$$f_1 + f_2 + f_3 = x - 1 + x^2 - x + 1 - x^2 = 0.$$

2. (6) Compute the Wronskian of e^{-x} , x , and x^2 . What can you conclude about linear independence?

$$W = \begin{vmatrix} e^{-x} & x & x^2 \\ -e^{-x} & 1 & 2x \\ e^{-x} & 0 & 2 \end{vmatrix} = e^{-x} \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} - x \begin{vmatrix} -e^{-x} & 2x \\ e^{-x} & 2 \end{vmatrix} + x^2 \begin{vmatrix} -e^{-x} & 1 \\ e^{-x} & 0 \end{vmatrix}$$

$$= 2e^{-x} + (2xe^{-x} + 2x^2e^{-x}) - x^2e^{-x}$$

$$= (2 + 2x + x^2)e^{-x}.$$

It's not zero, so the three functions are linearly independent.

3. (4,3) A homogeneous linear differential equation with constant coefficients has characteristic polynomial $(r-1)r^3(r^2-4r+5)^2(r-5)(r^2-4)$.

(a) Write down a general solution to this differential equation.

$$(r+2)(r-2)$$

$$\frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$y = Ae^x + B + Cx + Dx^2$$

$$+ Ee^{2x}\cos x + Fe^{2x}\sin x + Gxe^{2x}\cos x + Hxe^{2x}\sin x \\ + Ie^{5x} + Je^{-2x} + Ke^{2x}$$

- (b) Give an example of initial conditions for this equation which describe exactly one solution.

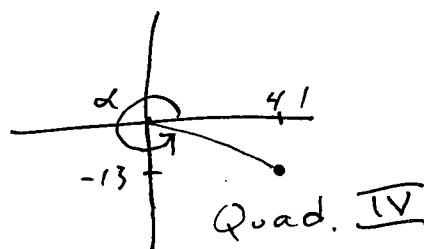
The equation is linear, of order 11. One possible answer:

$y(0) = 1$	$y^{(3)}(0) = 5$	$y^{(6)}(0) = 2$	$y^{(9)}(0) = 4$
$y'(0) = 4$	$y^{(4)}(0) = 7$	$y^{(7)}(0) = 1$	$y^{(10)}(0) = 0$
$y''(0) = 3$	$y^{(5)}(0) = 3$	$y^{(8)}(0) = 11$	

4. (6) Convert the function $x(t) = 41\cos(6t) - 13\sin(6t)$ into the form $x(t) = C\cos(\omega t - \alpha)$. Use an exact expression for α , rather than a decimal value.

$$C = \sqrt{A^2+B^2} \quad A = 41, \quad B = -13$$

$$= \sqrt{41^2 + 13^2}$$



$$x(t) = \sqrt{41^2 + 13^2} \cos\left(6t - \left(\tan^{-1}\left(\frac{-13}{41}\right) + 2\pi\right)\right)$$

5. (4,5) This problem concerns the differential equation $y''' - 4y'' + 6y' - 4y = 0$.

(a) Verify that $y_1 = 3e^{2x}$ is a solution.

$$y_1' = 6e^{2x}$$

$$y_1'' = 12e^{2x}$$

$$y_1''' = 24e^{2x}$$

$$\begin{aligned} y_1''' - 4y_1'' + 6y_1' - 4y_1 &= 24e^{2x} - 4(12e^{2x}) + 6(6e^{2x}) - 4(3e^{2x}) \\ &= (24 - 48 + 36 - 12)e^{2x} \\ &= 0. \end{aligned}$$

(b) Using part (a), find a general solution to the equation.

$$r^3 - 4r^2 + 6r - 4 = 0. \quad \text{One root is } r = 2.$$

$$\begin{array}{r} r^2 - 2r + 2 \\ \hline r-2 \sqrt{r^3 - 4r^2 + 6r - 4} \\ r^3 - 2r^2 \\ \hline -2r^2 + 6r \\ -2r^2 + 4r \\ \hline 2r - 4 \\ 2r - 4 \\ \hline 0 \end{array}$$

$$\text{So: } (r-2)(r^2-2r+2).$$

$$\text{Next: } \frac{z \pm \sqrt{4-8}}{2} = 1 \pm i$$

General solution!

$$y = Ae^{2x} + Be^{x}\cos x + Ce^{x}\sin x$$

6. (3,3,3,3) For the following differential equations, use the table below to determine the form you would use in finding a particular solution. Do not solve for any coefficients.

$f(x)$	y_p
$P_m = b_0 + b_1x + b_2x^2 + \dots + b_mx^m$	$x^s(A_0 + A_1x + A_2x^2 + \dots + A_mx^m)$
$a \cos kx + b \sin kx$	$x^s(A \cos kx + B \sin kx)$
$e^{rx}(a \cos kx + b \sin kx)$	$x^s e^{rx}(A \cos kx + B \sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1x + A_2x^2 + \dots + A_mx^m)e^{rx}$
$P_m(x)(a \cos kx + b \sin kx)$	$x^s[(A_0 + A_1x + \dots + A_mx^m) \cos kx + (B_0 + B_1x + \dots + B_mx^m) \sin kx]$

(a) $y'' - 9y = 3xe^{3x}$

$$\begin{aligned} r^2 - 9 &= 0 \\ r &= \pm 3 \end{aligned}$$

$$y_c = Ae^{3x} + Be^{-3x}$$

← duplication

try
$$y_p = x(A_0 + A_1x)e^{3x}$$

(b) $y'' + 2y' + y = 4e^{-x}$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r = -1, -1$$

$$y_c = Ae^{-x} + Bxe^{-x}$$

← duplication

try
$$y_p = x^2 Ae^{-x}$$

(c) $y'' + 16y = 2e^{4x}$

$$r^2 + 16 = 0$$

$$r = \pm 4i$$

$$y_c = A \cos(4x) + B \sin(4x)$$

try
$$y_p = Ae^{4x}$$

no duplication

(d) $y'' + 16y = 5 \cos(3x) + 2x \sin(2x)$

$$y_c = A \cos(4x) + B \sin(4x)$$

no duplication

try
$$y_p = A \cos(3x) + B \sin(3x)$$

$$+ (C + Dx) \cos(2x) + (E + Fx) \sin(2x)$$

7. (4.3) This problem concerns the differential equation $y'' + 16y = e^{3x}$.

(a) Using the method of undetermined coefficients, find a particular solution to the equation.

$$T_7 \quad y_p = Ae^{3x}$$

$$y_p' = 3Ae^{3x}$$

$$y_p'' = 9Ae^{3x}$$

$$y_p'' + 16y_p = 9Ae^{3x} + 16Ae^{3x} = e^{3x}$$

$$25Ae^{3x} = e^{3x}$$

$$A = \frac{1}{25}$$

$$\boxed{y_p = \frac{1}{25}e^{3x}}$$

(b) Find a general solution to the equation.

$$r^2 + 16 = 0, \quad r = \pm 4i$$

$$y_c = A\cos(4x) + B\sin(4x)$$

$$\boxed{y = A\cos(4x) + B\sin(4x) + \frac{1}{25}e^{3x}}$$

Extra Credit (3) Use Euler's formula $e^{ix} = \cos x + i \sin x$ to derive the formula for $\cos(A - B)$.

$$e^{i(A-B)} = \cos(A - B) + i \sin(A - B)$$

$$e^{iA} e^{i(-B)} = (\cos A + i \sin A)(\underbrace{\cos(-B)}_{\cos B} + i \sin(-B))$$

$$= \cos A \cos B + i \sin A \cos B + i \cos A \sin(-B) + \underbrace{i^2 \sin A \sin(-B)}_{= \sin A \sin B}$$

Take real parts:

$$\boxed{\cos(A - B) = \cos A \cos B + \sin A \sin B}.$$