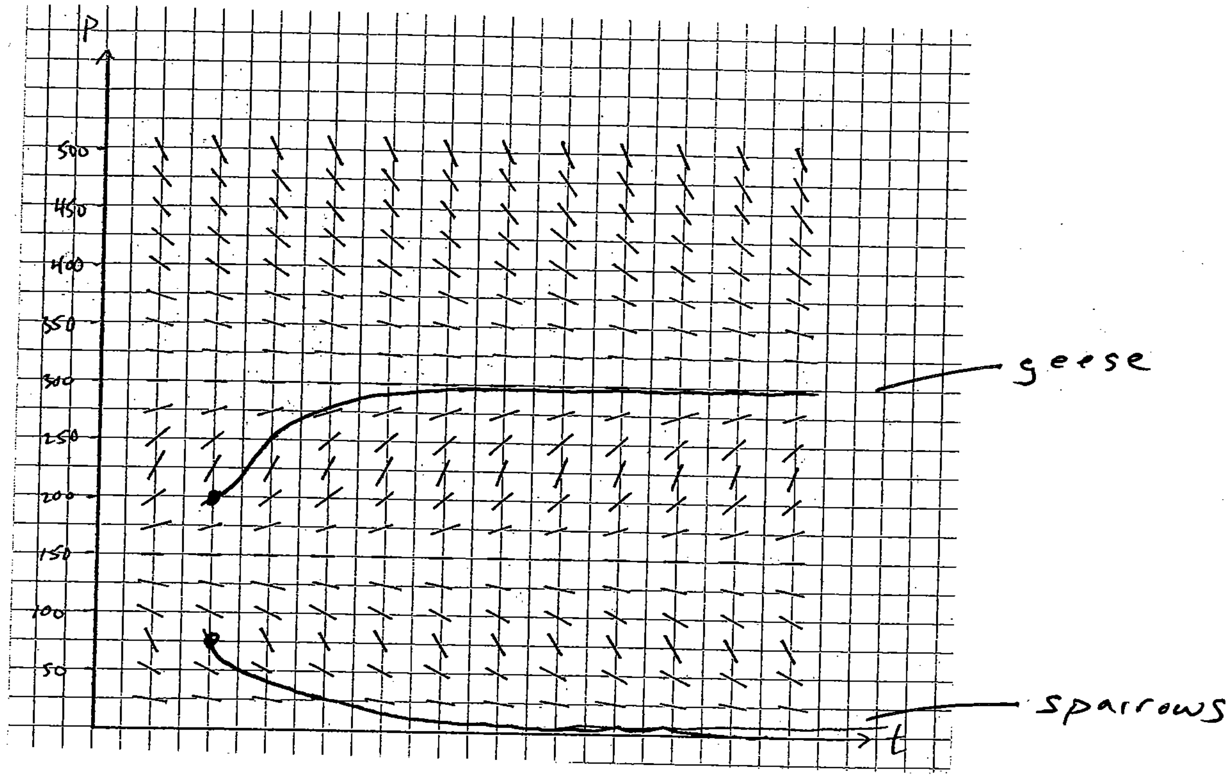
1. (9 points) On a certain island, bird populations satisfy a differential equation whose slope field is shown below. Each species has its own population function, but they all satisfy the same differential equation.



At some point in time the sparrow population is 75 and the goose population is 200.

(a) Describe in words what happens to these birds over time. Be as specific as you can.

(b) What must be true if a bird species is to succeed on this island? State your condition carefully.

(a) the sparrow population will decrease and eventually die out. the goose population will rise and level off at 300.

(b) a species must have at least 150 members to survive.

2. (7 points) Solve the initial value problem $\frac{dy}{dx} = 4x^3y - y,\ \ y(1) = 30.$

$$\frac{dy}{dx} = (4x^3 - 1)y$$

$$\int \frac{dy}{y} = \int (4x^3 - 1)\,dx$$

$$\ln|y| = x^4 - x + c$$

$$|y| = e^{x^4-x}e^c$$

$$y = \pm e^c e^{x^4-x} = Ae^{x^4-x}$$

$$30 = Ae^{1-1} = A$$

$$\boxed{y = 30e^{x^4-x}}$$

3. (7 points) Find the integrating factor for the linear equation $2xy' - 3y = 9x^3$. (Do not solve the equation.)

$$2x\frac{dy}{dx} - 3y = 9x^3$$

$$\frac{dy}{dx} \underbrace{- \frac{3}{2x}}_{P(x)} y = \underbrace{3x^2}_{Q(x)}$$

$$\rho(x) = e^{\int \frac{-3}{2x}dx} = e^{-\frac{3}{2}\ln|x|} = \boxed{x^{-3/2}}$$

4. (9 points) Recall Newton's law of heating and cooling: the rate of change of the temperature of an object is proportional to the difference between its temperature and the ambient temperature.

A pitcher of buttermilk initially at 25°C is to be cooled by setting it on the front porch, where the temperature is 0°C. Suppose that the temperature has dropped to 15°C after 15 minutes.

Find the temperature function $T(t)$.

$$\frac{dT}{dt} = k(T-A)$$
$$= kT$$

$A = 0$

$T(0) = 25$

$T(15) = 15$

$$\int \frac{dT}{T} = \int k\,dt$$

$$\ln|T| = kt + C$$

$$|T| = e^{kt} e^{C}$$

$$T = \pm e^{C} e^{kt} = Ae^{kt}$$

$$25 = Ae^{0} = A$$

$$T = 25e^{kt}$$

$$15 = 25e^{k(15)}$$

$$\frac{3}{5} = e^{15k}$$

$$\ln(3/5) = 15k$$

$$k = \frac{1}{15}\ln(3/5)$$

$$\boxed{T(t) = 25e^{\frac{\ln(3/5)}{15}t}}$$

5. (7 points) Solve the initial value problem $3x - xy' = 2y, \ y(1) = 0$.

$$xy' = 3x - 2y$$

$$y' + \frac{2}{x}y = 3 \quad \text{linear}$$

$$\rho(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$$

$$\underbrace{x^2y' + x^2\frac{2}{x}y}_{\frac{d}{dx}(x^2y)} = 3x^2$$

$$x^2y = \int 3x^2 dx = x^3 + C$$

$$y = x + \frac{C}{x^2}$$

$$0 = 1 + \frac{C}{1}$$

$$C = -1$$

$$\boxed{y = x - \frac{1}{x^2}}$$

6. (7 points) Use substitution to solve the second order equation $(1+x)y'' = 4y'$.

$p = y'$

$p' = y''$ (all derivatives with respect to x)

$$(1+x)\frac{dp}{dx} = 4p$$

$$\int \frac{dp}{p} = \int \frac{4\,dx}{1+x}$$

$$\ln|p| = 4\ln|1+x| + C$$

$$p = \pm e^C(1+x)^4 = A(1+x)^4$$

$$y' = p$$

$$y = \int A(1+x)^4 dx = \boxed{\frac{A}{5}(1+x)^5 + B}$$

7. (9 points) The equation $2xy' + y^3e^{-2x} = 2xy$ is either a homogeneous equation or a Bernoulli equation. Find the solution by substituting $v = y/x$ or $v = y^{1-n}$ accordingly.

$$2xy' - 2xy = -e^{-2x}y^3$$

$$\underbrace{y' - y}_{P(x)} = \underbrace{\frac{-e^{-2x}}{2x}}_{Q(x)} y^3 \qquad n = 3$$

$$v = y^{-2} \Rightarrow y = v^{-1/2} \quad \text{so} \quad \frac{dy}{dx} = \frac{-1}{2} v^{-3/2} \frac{dv}{dx}.$$

substitute:

$$\frac{-1}{2} v^{-3/2} \frac{dv}{dx} - v^{-1/2} = \frac{-e^{-2x}}{2x} v^{-3/2}$$

$$\frac{dv}{dx} + \underbrace{2}_{P}v = \underbrace{\frac{e^{-2x}}{x}}_{Q} \qquad \text{linear}$$

$$\rho(x) = e^{\int 2\,dx} = e^{2x}$$

$$\underbrace{e^{2x}\frac{dv}{dx} + e^{2x}\cdot 2v}_{\frac{d}{dx}(e^{2x}v)} = \frac{1}{x}$$

$$e^{2x}v = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$v = e^{-2x}(\ln|x| + C)$$

$$y = \left[e^{-2x}(\ln|x| + C)\right]^{-1/2} = \boxed{\frac{e^{x}}{\sqrt{\ln|x| + C}}}$$