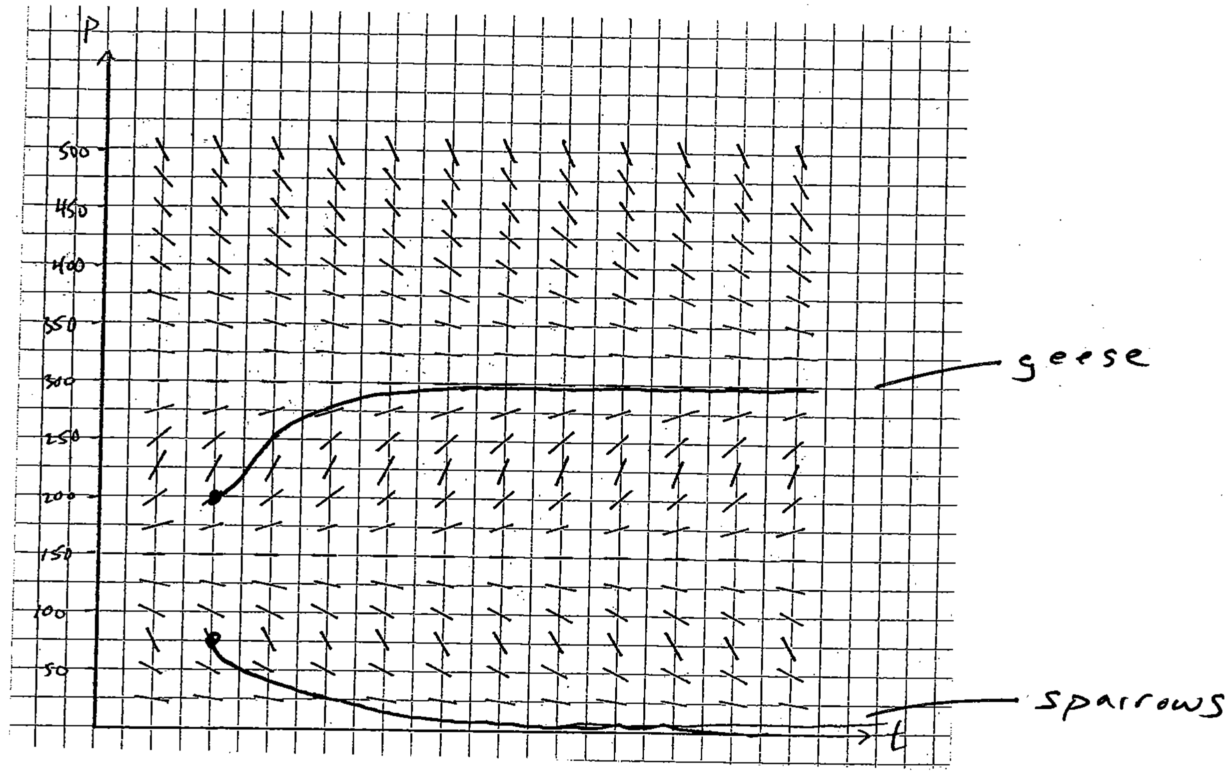
1. (9 points) On a certain island, bird populations satisfy a differential equation whose slope field is shown below. Each species has its own population function, but they all satisfy the same differential equation.



At some point in time the sparrow population is 75 and the goose population is 200.

(a) Describe in words what happens to these birds over time. Be as specific as you can.

(b) What must be true if a bird species is to succeed on this island? State your condition carefully.

(a) the sparrow population will decrease and eventually die out. the goose population will rise and level off at 300.

(b) a species must have at least 150 members to survive.

2. (9 points) Find the integrating factor for $x\frac{dy}{dx} - 2y = 3x$. Proceed to solve the equation (showing all steps), but stop when you are required to do an integral.

$$\frac{dy}{dx} \underbrace{- \frac{2}{x}}_{P(x)} y = \underbrace{3}_{Q(x)}$$

$$\rho(x) = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = \boxed{x^{-2}}$$

$$\underbrace{x^{-2}\frac{dy}{dx} - x^{-2}\frac{2}{x}y}_{\frac{d}{dx}(x^{-2}y)} = 3x^{-2}$$

$$\boxed{x^{-2}y = \int 3x^{-2}\,dx}$$

3. (9 points) Use substitution to transform the equation $\frac{dy}{dx} = (4x+y)^2$ into a separable equation. Separate the equation, but do not solve it.

$$v = 4x + y$$

$$y = v - 4x \quad , \quad \frac{dy}{dx} = \frac{dv}{dx} - 4$$

sub. to get:

$$\frac{dv}{dx} - 4 = v^2$$

$$\frac{dv}{dx} = v^2 + 4$$

$$\boxed{\int \frac{dv}{v^2+4} = \int dx}$$

4. (9 points) Solve the initial value problem $x\frac{dy}{dx} - y = 2x^2y,\ \ y(1) = 1$.

$$x\frac{dy}{dx} = 2x^2y + y = (2x^2+1)y$$

$$\int \frac{dy}{y} = \int \frac{2x^2+1}{x}\,dx = \int 2x\,dx + \int \frac{1}{x}\,dx$$

$$\ln|y| = x^2 + \ln|x| + C$$

$$|y| = e^{x^2} e^{\ln|x|} e^{C}$$

$$y = \pm e^{C}|x|e^{x^2}$$

$$y = A|x|e^{x^2}, \quad A \neq 0.$$

$y(1) = 1$:

$$1 = Ae$$

$$A = \frac{1}{e}$$

$$\boxed{y = |x|e^{x^2-1}}$$

5. (9 points) Use substitution to solve the second order equation $(1+x)y'' = 4y'$.

Let $p(x) = \frac{dy}{dx} = y'$

$$p'(x) = \frac{d^2y}{dx^2} = y''$$

$$(1+x)\frac{dp}{dx} = 4p$$

$$\int \frac{dp}{p} = \int \frac{4}{1+x}\,dx$$

$$\ln|p| = 4\ln|1+x| + C$$

$$p = \pm e^C (1+x)^4$$

$$\frac{dy}{dx} = A(1+x)^4$$

$$y = \int A(1+x)^4\,dx$$

$$\boxed{y = \frac{A}{5}(1+x)^5 + B}$$

6. (10 points) Solve the Bernoulli equation $2xy' + y^3e^{-2x} = 2xy$ by substituting $v = y^{1-n}$ for the appropriate value of n.

$$2xy' - 2xy = -e^{-2x}y^3$$

$$y' \underbrace{-\,y}_{P} = \underbrace{\frac{-e^{-2x}}{2x}}_{Q}\, y^3 \quad n=3$$

$$v = y^{-2}$$

$$y = v^{-1/2} \qquad \frac{dy}{dx} = \frac{-1}{2} v^{-3/2} \frac{dv}{dx}$$

sub:

$$\frac{-1}{2} v^{-3/2} \frac{dv}{dx} - v^{-1/2} = \frac{-e^{-2x}}{2x} v^{-3/2}$$

$$\frac{dv}{dx} + \underbrace{2}_{P} v = \underbrace{\frac{e^{-2x}}{x}}_{Q}$$

$$\rho(x) = e^{\int 2\,dx} = e^{2x}$$

$$\underbrace{e^{2x}\frac{dv}{dx} + e^{2x}\,2v}_{\frac{d}{dx}(e^{2x}v)} = e^{2x}\frac{e^{-2x}}{x}$$

$$e^{2x}v = \int \frac{1}{x}\,dx = \ln|x| + C$$

$$v = e^{-2x}(\ln|x| + C)$$

$$y = \left[e^{-2x}(\ln|x| + C)\right]^{-1/2} = \boxed{\frac{e^{x}}{\sqrt{\ln|x| + C}}}$$