1. Let $f(x,y) = y^2\cos(x) + 4(x^3+1)y$. Use linear approximation (or a tangent plane) at an appropriate point (a,b) to estimate $f(0.02, 2.01)$.

- What is the appropriate point (a,b)?
- What is the linear approximation or tangent plane (your choice) at that point?
- Estimate $f(0.02, 2.01)$.

use $\boxed{(a,b) = (0,2)}$ Note, $f(0,2) = 12$.

now, need $f_x(0,2)$ and $f_y(0,2)$.

$$f_x(x,y) = -y^2\sin x + 12x^2y, \quad f_x(0,2) = 0$$

$$f_y(x,y) = 2y\cos x + 4(x^3+1), \quad f_y(0,2) = 8$$

linearization at $(0,2)$ is

$$L(x,y) = 0(x-0) + 8(y-2) + 12$$

$$\boxed{L(x,y) = 8y - 4}$$

(tangent plane: $z = 8y - 4$)

$$f(0.02, 2.01) \approx L(0.02, 2.01)$$

$$= 8(2.01) - 4$$

$$= 16.08 - 4$$

$$= \boxed{12.08}$$

2(a) Write down an integral which computes the length of the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \tan t \rangle$, $0 \le t \le \pi/4$. [Do not compute the integral or proceed any further.]

$$\vec{r}\,'(t) = \langle \cos t, -\sin t, \sec^2 t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\cos^2 t + \sin^2 t + \sec^4 t} = \sqrt{1 + \sec^4 t}$$

arc length is $\boxed{\int_0^{\pi/4} \sqrt{1 + \sec^4 t}\, dt}$

2(b) The function $f(x, y) = 2 + x^3 + y^3 - 3xy$ has critical points at $(1, 1)$ and $(0, 0)$. What does the second derivatives test allow you to conclude about f at these points?

$f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$

$f_{xx}(x,y) = 6x$ $\quad$ $f_{yy}(x,y) = 6y$

$f_{xy}(x,y) = -3$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 36xy - 9$$

So At $(0,0)$, $D = -9$, $f_{xx} = 0$

$\hookrightarrow$ $\boxed{\text{f has a saddle point at } (0,0)}$

At $(1,1)$, $D = 27$, $f_{xx} = 6$

$\hookrightarrow$ $\boxed{\text{f has a local minimum at } (1,1)}$

3(a) Suppose we want to find the point on the surface $x^2yz^3 = 4$ that is closest to the origin. Use Lagrange multipliers to write down a system of equations whose solution will include this point. [Do not solve the system or proceed any further.]

Let $f(x,y,z)$ = squared distance to origin $= x^2+y^2+z^2$.

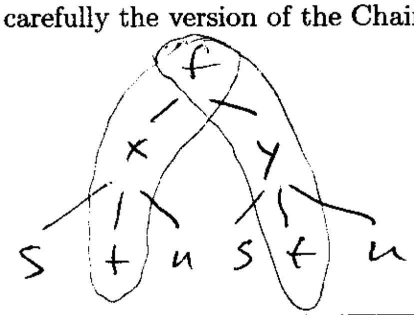
Constraint: $\underbrace{x^2yz^3}_{g(x,y,z)} = 4$

$$\nabla f = \langle 2x, 2y, 2z \rangle, \quad \nabla g = \langle 2xyz^3, x^2z^3, 3x^2yz^2 \rangle$$

system:

$$\boxed{\begin{aligned} 2x &= \lambda \cdot 2xyz^3 \\ 2y &= \lambda \cdot x^2z^3 \\ 2z &= \lambda \cdot 3x^2yz^2 \\ x^2yz^3 &= 4 \end{aligned} \quad \text{in the variables } x, y, z, \lambda}$$

3(b) Let $f(x,y) = e^{xy}$. Evaluate $\frac{\partial f}{\partial t}$ at $(s,t,u) = (2,-1,3)$ where $x = tu$ and $y = t - s$. State carefully the version of the Chain Rule that you use.



Chain Rule:

$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

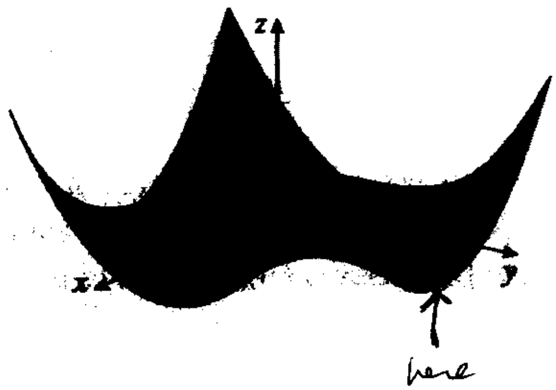
We have $\frac{\partial f}{\partial x} = ye^{xy}$, $\frac{\partial f}{\partial y} = xe^{xy}$

$$\frac{\partial x}{\partial t} = u, \quad \frac{\partial y}{\partial t} = 1$$

when $(s,t,u) = (2,-1,3)$, $x = -3$ and $y = -3$

So $$\frac{\partial f}{\partial t} = \left((-3)e^{(-3)(-3)}\right)\cdot 3 + \left((-3)e^{(-3)(-3)}\right)\cdot 1$$
$$= \boxed{-12e^9}$$

4(a) For the function whose graph is shown below, determine whether each of the following derivatives is positive, negative, or zero: $f_x(0,9)$, $f_y(0,9)$, $f_{xx}(0,9)$, $f_{yy}(0,9)$, $f_{xy}(0,9)$. The scale in the picture is such that the spacing between the curves represents one unit. [Locate the relevant point first, then answer the question.]



$f_x(0,9) = 0$ (slope in x-dir.)

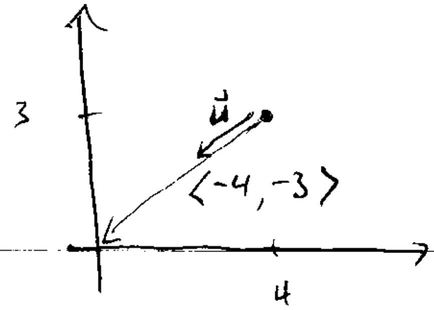
$f_y(0,9) < 0$ (slope in y-dir.)

$f_{xx}(0,9) > 0$ (concavity)

$f_{yy}(0,9) < 0$ (concavity)

$f_{xy}(0,9) = 0$ (f_x remains 0 as you move in y-dir.)

4(b) Let $f(x,y) = 2xy - y^2$. Find the directional derivative of $f(x,y)$ at $(4,3)$ in the direction of the origin.



Direction of origin $= \dfrac{\langle -4,-3\rangle}{|\langle -4,-3\rangle|}$

$\vec{u} = \langle \frac{-4}{5}, \frac{-3}{5} \rangle$.

Next, $\nabla f = \langle 2y, 2x-2y \rangle$, so $\nabla f(4,3) = \langle 6, 2 \rangle$.

$D_{\vec{u}} f(4,3) = \nabla f(4,3) \cdot \vec{u} = \langle 6,2 \rangle \cdot \langle \frac{-4}{5}, \frac{-3}{5} \rangle$

$= \frac{-24}{5} + \frac{-6}{5} = \frac{-30}{5} = \boxed{-6}$