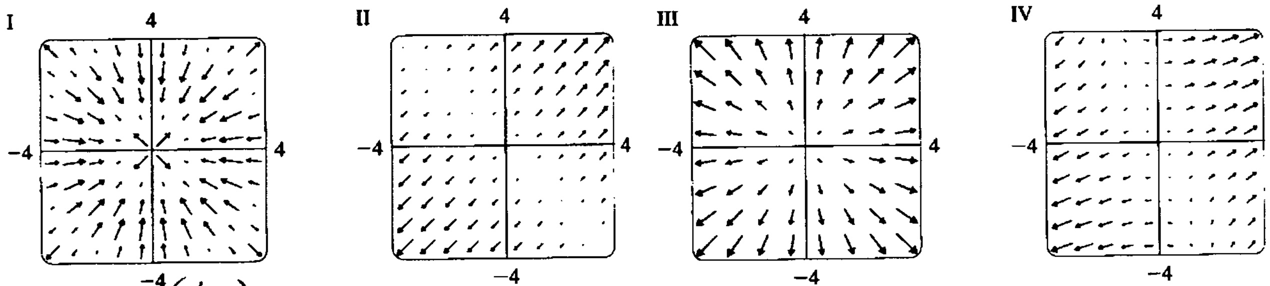
1(a) Match the functions f with the plots of their gradient vector fields labeled I–IV.

(i) $f(x, y) = x^2 + y^2$
(ii) $f(x, y) = x(x + y)$
(iii) $f(x, y) = (x + y)^2$
(iv) $f(x, y) = \sin\sqrt{x^2 + y^2}$



I: (iv) II: (iii) III: (i) IV: (ii)

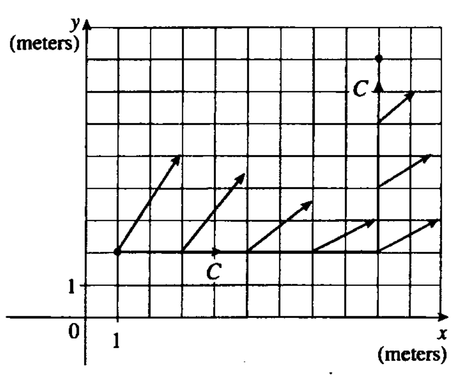
(i) $\langle 2x, 2y \rangle$

(ii) $\langle 2x+y, x \rangle$

(iii) $\langle 2x+2y, 2x+2y \rangle$

(iv) $\left\langle \cos(\sqrt{x^2+y^2})\frac{x}{\sqrt{x^2+y^2}}, \cos(\sqrt{x^2+y^2})\frac{y}{\sqrt{x^2+y^2}} \right\rangle$

(b) A path C and a vector field $\mathbf{F}$ are shown below. Estimate the numerical value of $\int_C \mathbf{F} \cdot d\mathbf{r}$. [You can ignore the units.]



Let C_1 = bottom path
C_2 = right path.

Along C_1, $\vec{F}\cdot\vec{T}$ is 2 everywhere

so $\int_{C_1} \vec{F}\cdot\vec{T}\,ds = 2\int_{C_1} ds$
$= 2 \cdot \text{length}(C_1)$

Along C_2, $\vec{F}\cdot\vec{T}$ is 1 everywhere

so $\int_{C_2} \vec{F}\cdot\vec{T}\,ds = \int_{C_2} ds = \text{length}(C_2)$

$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds = 2(8) + 6 = \boxed{22}$

2. Find the maximum of $f(x, y) = 2x + y - 3xy$ on the unit square $0 \le x, y \le 1$.

(a) What are the critical points?

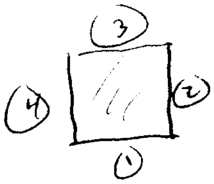
(b) What other points are relevant? Explain.

(c) Find the maximum.

(a) $f_x = 2 - 3y$
$f_y = 1 - 3x$ set $= 0 \Rightarrow y = \frac{2}{3}$ and $x = \frac{1}{3}$.

$\boxed{\text{Critical Points: } (\tfrac{1}{3}, \tfrac{2}{3})}$

(b) must also check all points on the boundary.



1. $f(x,0) = 2x, \quad 0 \le x \le 1$
2. $f(1,y) = 2 - 2y, \quad 0 \le y \le 1$
3. $f(x,1) = 1 - x, \quad 0 \le x \le 1$
4. $f(0,y) = y, \quad 0 \le y \le 1$

These are linear, so the max. will be at an endpoint.

$\boxed{\text{In all, check the four corners: } (0,0), (0,1), (1,0), (1,1).}$

(c) Evaluate:

$f(0,0) = 0$

$f(0,1) = 1$

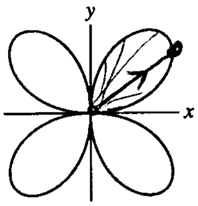
$f(1,0) = 2$

$f(1,1) = 0$

$f(\frac{1}{3}, \frac{2}{3}) = \frac{2}{3} + \frac{2}{3} - \frac{6}{9} = \frac{2}{3}.$

$\boxed{\text{Maximum is } 2, \text{ at } (1,0).}$

3(a) Find the area of one loop of the curve $r = \sin(2\theta)$, shown in the figure below. [What is the relevant range of θ values?]



region: $0 \le \theta \le \pi/2$, $0 \le r \le \sin(2\theta)$

$$\text{Area} = \int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = \int_0^{\pi/2} \tfrac{1}{2}r^2\Big|_0^{\sin 2\theta} d\theta$$

$$= \int_0^{\pi/2} \tfrac{1}{2}\sin^2(2\theta)\,d\theta$$ use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$= \int_0^{\pi/2} \tfrac{1}{4}(1-\cos(4\theta))\,d\theta$$

$$= \tfrac{1}{4}\theta - \tfrac{1}{16}\sin(4\theta)\Big|_0^{\pi/2}$$

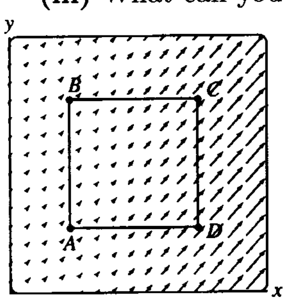
$$= \boxed{\frac{\pi}{8}} - 0 + 0$$

3(b) The figure below shows a vector field $\mathbf{F} = \langle P, Q\rangle$. Let C be the square, traversed in the counter-clockwise direction (ADCB).

(i) Is the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ positive, negative, or zero? Explain.

(ii) Is this vector field conservative?

(iii) What can you say about the function $Q_x - P_y$ inside the square? Explain.



(i) $\boxed{\text{Positive}}$. AD and CB cancel, but $\int_{DC} F\cdot dr > \int_{AB} F\cdot dr$.

(ii) $\boxed{\text{No}}$, since C is closed and $\int_C F\cdot dr \neq 0$.

(iii) $Q_x - P_y$ is, on average, $\boxed{\text{positive}}$ inside the square. This is because

$$\iint_{\substack{\text{square}\\\text{region}}} (Q_x - P_y)\,dA = \int_C F\cdot dr > 0.$$

Green's Theorem

4(a) Let $\mathbf{F}(x,y,z) = \langle x^2yz, xy^2z, xyz^2\rangle$. Use the Divergence Theorem to calculate the surface integral $\int_S \mathbf{F}\cdot\mathbf{n}\,dS$, where S is the surface of the box bounded by the coordinate planes and the planes $x=2$, $y=3$, and $z=4$.

By divergence theorem, we need to compute $\iiint_E \operatorname{div}(\vec{F})\,dV$ where E is the region inside the box.

$\operatorname{div}(\vec{F}) = 2xyz + 2xyz + 2xyz = 6xyz$.

So:
$$\int_0^4\int_0^3\int_0^2 6xyz \, dx\,dy\,dz$$
$$= 6\int_0^2 x\,dx \cdot \int_0^3 y\,dy \cdot \int_0^4 z\,dz$$
$$= 6\left[\tfrac{1}{2}x^2\Big|_0^2\right]\left[\tfrac{1}{2}y^2\Big|_0^3\right]\left[\tfrac{1}{2}z^2\Big|_0^4\right]$$
$$= 6(2)\left(\tfrac{9}{2}\right)(8) = \boxed{432}$$

(b) Find the equation of the tangent plane to the graph of $f(x,y) = e^{-xy}\cos y$ at the point $(\pi, 0, 1)$.

equation:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

where $(a,b) = (\pi, 0)$

$f_x = -ye^{-xy}\cos y$, $f_x(\pi,0) = 0$

$f_y = e^{-xy}(-\sin y) + (-x)e^{-xy}\cos y$, $f_y(\pi,0) = -\pi$

$$\boxed{z - 1 = -\pi y}$$

5(a) Let $\mathbf{F}(x,y,z) = \langle xy, -z, ye^{xz}\rangle$ and let S be the parallelogram with parametrization $\mathbf{r}(u,v) = \langle u+v, u-v, 1+2u+v\rangle$, with domain $0 \le u \le 2$, $0 \le v \le 1$, and with the upward orientation for $\mathbf{n}$. Express $\int_S \mathbf{F}\cdot d\mathbf{S}$ as an ordinary double integral in u and v. Do not solve the integral.

$\vec{r}_u = \langle 1, 1, 2\rangle$ $\quad \vec{r}_u \times \vec{r}_v = \langle 3, 1, -2\rangle$, $\dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|} = \dfrac{1}{\sqrt{14}}\langle 3,1,-2\rangle$

$\vec{r}_v = \langle 1, -1, 1\rangle$ $\quad$ Since $\vec{n}$ points upward (i.e. positive z-cpnt)

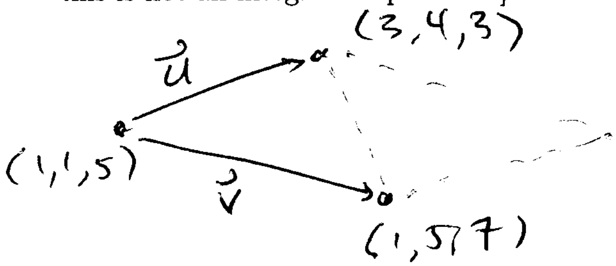
it must be $\dfrac{-\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$.

So

$$\int_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(-\vec{r}_u\times\vec{r}_v)\,dA, = \iint \langle (u+v)(u-v), -1-2u-v, (u-v)e^{(u+v)(1+2u+v)}\rangle \cdot \langle -3,-1,2\rangle\, dA$$

$$= \boxed{\int_0^2\int_0^1 -3(u^2-v^2)+1+2u+v+2(u-v)e^{(u+v)(1+2u+v)}\,dv\,du}$$

5(b) Find the area of the triangle with corners at the points $(1,1,5)$, $(3,4,3)$, and $(1,5,7)$. [Hint: this is not an integration problem.]



$\vec{u} = \langle 2, 3, -2\rangle$

$\vec{v} = \langle 0, 4, 2\rangle$

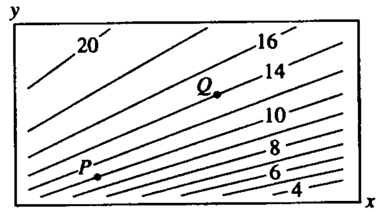
$\vec{u}\times\vec{v} = \langle 14, -4, 8\rangle$

$$\text{Area} = \frac{1}{2}|\vec{u}\times\vec{v}| = \frac{1}{2}\sqrt{14^2+16+64}$$

$$= \sqrt{7^2+4+16}$$

$$= \boxed{\sqrt{69}}$$

6(a) The function $f(x, y)$ has level curves shown below.
(i) Is f_x larger at P or at Q? What about f_y? Explain.
(ii) At P is f_{xy} positive, negative, or zero? Explain.



(i) f_x is negative at P and Q, but steeper at P.
So $\boxed{f_x(Q) > f_x(P)}$.

f_y is positive at P and Q, and steeper at P. $\boxed{f_y(P) > f_y(Q)}$

(ii) At P, $f_x < 0$. As you move in y-direction, the x-slope gets less steep, i.e. it increases (toward 0).
So f_x is increasing.
$\boxed{(f_x)_y > 0}$

6(b) Find the cosine of the angle between the planes $x + y + z = 4$ and $-3x + y - 2z = 2$.

This is the same as the cosine of the angle between their normal vectors $\vec{n}_1$ and $\vec{n}_2$.

$\vec{n}_1 = \langle 1, 1, 1 \rangle$ and $\vec{n}_2 = \langle -3, 1, -2 \rangle$,

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{-3+1-2}{\sqrt{3}\sqrt{14}}$$

$$= \boxed{\frac{-4}{\sqrt{42}}}$$