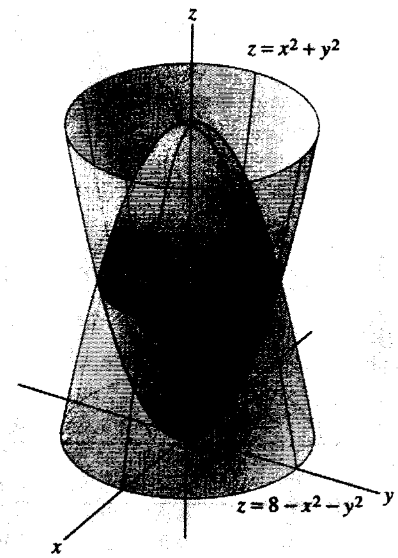
1. Use cylindrical coordinates to find the volume of the region between the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$:

(a) What is the appropriate region D in the xy-plane?

(b) Find the volume.



The circle of intersection satisfies $x^2+y^2 = 8 - x^2 - y^2$

$$2x^2+2y^2 = 8$$

$$x^2+y^2 = 4$$

so $\boxed{D = \text{disk of radius 2 centered at the origin.}}$

$$\text{Volume} = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \left[rz \Big|_{r^2}^{8-r^2} \right] dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 8r - 2r^3 \, dr\, d\theta$$

$$= \int_0^{2\pi} \left[4r^2 - \tfrac{1}{2} r^4 \Big|_0^2 \right] d\theta$$

$$= 2\pi (16 - 8)$$

$$= \boxed{16\pi}$$

2. Use Lagrange multipliers to find the minimum and maximum values of $f(x, y) = x^2y$ on the ellipse $4x^2 + 9y^2 = 36$.

$f(x,y) = x^2y$, $g(x,y) = 4x^2 + 9y^2 = 36$

$\nabla f = \langle 2xy, x^2 \rangle$ $\nabla g = \langle 8x, 18y \rangle$

system: ① $2xy = \lambda \cdot 8x$

② $x^2 = \lambda \cdot 18y$

③ $4x^2 + 9y^2 = 36$

① gives $x = 0$ or $y = 4\lambda$.

If $x = 0$, ③ gives $9y^2 = 36$, $y^2 = 4$, $y = \pm 2$

so $(0, \pm 2)$.

If $x \neq 0$, $\lambda = \frac{1}{4}y$ so ② gives $x^2 = \frac{18}{4}y^2$ ④

and ③ gives $18y^2 + 9y^2 = 36$

$27y^2 = 36 \Rightarrow y^2 = \frac{4}{3} \overset{④}{\Rightarrow} x^2 = 6$

so $(\pm\sqrt{6}, \pm\frac{2}{\sqrt{3}})$.

Evaluate: $f(0, \pm 2) = 0$

$f(\pm\sqrt{6}, \frac{2}{\sqrt{3}}) = \frac{12}{\sqrt{3}}$ ← max. value

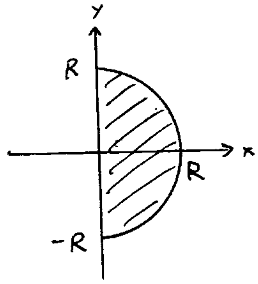
$f(\pm\sqrt{6}, \frac{-2}{\sqrt{3}}) = \frac{-12}{\sqrt{3}}$ ← min. value

3. Let D be the region inside the disk of radius R and to the right of the y-axis, as shown below.

(a) What is the area of D?

(b) Find $\iint_D x\,dA$.

(c) What is the average value of the x-coordinates of the points in D?



$$\boxed{\text{Area} = \tfrac{1}{2}\pi R^2}$$

$$\iint_D x\,dA = \int_{-\pi/2}^{\pi/2}\int_0^R r\cos\theta \, r \, dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta \cdot \int_0^R r^2\,dr$$

$$= \Big[\underbrace{\sin\theta \Big|_{-\pi/2}^{\pi/2}}_{1-(-1)}\Big]\Big[\tfrac{1}{3}r^3\Big|_0^R\Big] = \boxed{\tfrac{2}{3}R^3}.$$

Average value of x-coordinate is

$$\frac{1}{\text{Area}}\iint_D f(x,y)\,dA \quad \text{where } f(x,y) = x$$

$$= \frac{2}{3}R^3 \Big/ \frac{\pi}{2}R^3 = \boxed{\frac{4}{3\pi}R}$$

4(a) Express the integral

$$\iiint_E xyz\, dV$$

in spherical coordinates, where the region E is the portion of the unit ball that lies in the first octant ($x, y, z \geq 0$). [Do not evaluate the integral.]

First octant is given by: $0 \leq \varphi \leq \pi/2$, $0 \leq \theta \leq \pi/2$

$$\iiint_E xyz\, dV =$$

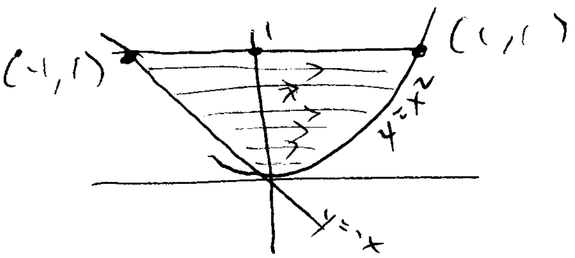
$$\boxed{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)(\rho\cos\varphi)\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}$$

(b) Sketch the region, and change the order of integration for

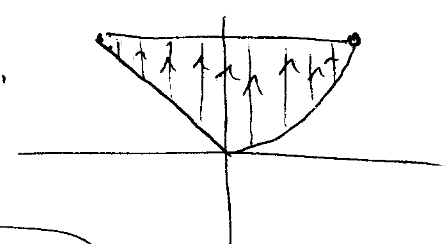
$$\int_0^1 \int_{-y}^{\sqrt{y}} f(x,y)\, dx\, dy.$$

region: left curve: $x = -y$ right curve: $x = \sqrt{y}$

$y = x^2$



reverse order:



$$\boxed{\int_{-1}^{0}\int_{-x}^{1} f(x,y)\, dy\, dx + \int_0^1\int_{x^2}^{1} f(x,y)\, dy\, dx}$$