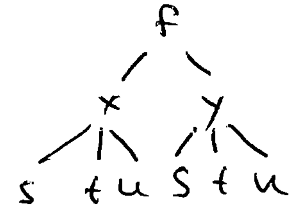
1(a) Let $f(x,y) = e^{xy}$. Evaluate $\frac{\partial f}{\partial t}$ at $(s,t,u) = (-1,3,5)$ where $x = st$ and $y = u - t$. State carefully the version of the Chain Rule that you use.



$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

$\frac{\partial f}{\partial x} = ye^{xy}$ $\quad \frac{\partial x}{\partial t} = s$

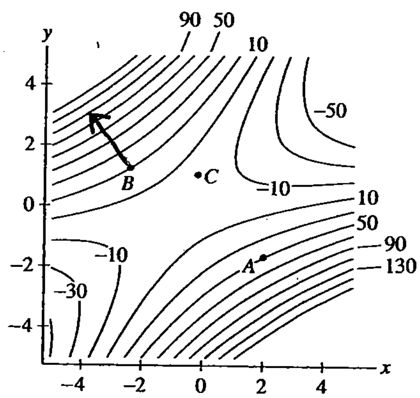
$\frac{\partial f}{\partial y} = xe^{xy}$ $\quad \frac{\partial y}{\partial t} = -1$

At $(s,t,u) = (-1,3,5)$
$(x,y) = (-3,2)$

So $\frac{\partial f}{\partial x} = 2e^{-6}$, $\frac{\partial f}{\partial y} = -3e^{-6}$, $\frac{\partial x}{\partial t} = -1$, $\frac{\partial y}{\partial t} = -1$

So $\frac{\partial f}{\partial t} = 2e^{-6}(-1) + (-3e^{-6})(-1) = \boxed{e^{-6}}$

1(b) The figure shows a contour map of $f(x,y)$.
(i) Estimate f_x and f_y at point A.
(ii) At point B, in which direction does f increase most rapidly?
(iii) At which of A, B, or C is f_y smallest?
(iv) At A, is f_{yx} positive, negative, or zero? Explain briefly.



(i) $f_x(A) \approx 20$, $f_y \approx -40$

(ii) in direction indicated by arrow

(iii) f_y smallest at $\boxed{A}$ (positive at B, near 0 at C)

(iv) moving in x-direction, contour spacing gets closer → steeper. So f_y decreases (since it's negative).

So $\boxed{f_{yx} \text{ is negative}}$.

2. Let $f(x, y) = \dfrac{x^2}{y^2+1}$. Use linear approximation (or a tangent plane) at an appropriate point (a, b) to estimate $f(4.01, 0.98)$.

- What is the appropriate point (a, b)?
- What is the linear approximation or tangent plane (your choice)?
- Estimate $f(4.01, 0.98)$.

Use $(a,b) = \boxed{(4,1)}$.

$f_x(x,y) = \dfrac{2x}{y^2+1}$, $f_x(4,1) = 4$.

$f_y(x,y) = \dfrac{-x^2}{y^2+1} \cdot 2y$, $f_y(4,1) = -8$

$L(x,y) = f(4,1) + f_x(4,1)(x-4) + f_y(4,1)(y-1)$

$$\boxed{L(x,y) = 8 + 4(x-4) - 8(y-1)}$$

now,

$f(4.01, 0.98) \approx L(4.01, 0.98)$

$= 8 + 4(0.01) - 8(-.02)$

$= 8 + .04 + .16$

$= \boxed{8.20}$

3. Find the critical points of the function $f(x, y) = (x^2 + y^2)e^{-x}$. Then analyze them using the Second Derivatives Test. What can you conclude about the behavior of f at these points?

$$f_x(x,y) = 2xe^{-x} - (x^2+y^2)e^{-x}$$
$$= e^{-x}(2x - x^2 - y^2) = 0 \Rightarrow 2x - x^2 - y^2 = 0.$$

$$f_y(x,y) = 2ye^{-x} = 0 \Rightarrow \underline{y=0}.$$

put into first eqn: $2x - x^2 = 0$

$x(2-x) = 0$

$\underline{x=0}$ or $\underline{x=2}$

<u>Critical Points:</u> $\boxed{(0,0), (2,0)}$

$$f_{xx}(x,y) = e^{-x}(2-2x) - e^{-x}(2x - x^2 - y^2)$$

$\underline{f_{xx}(0,0) = 2}$, $\underline{f_{xx}(2,0) = -2e^{-2}}$

$$f_{yy}(x,y) = 2e^{-x}$$

$\underline{f_{yy}(0,0) = 2}$, $\underline{f_{yy}(2,0) = 2e^{-2}}$

$$f_{yx}(x,y) = -2ye^{-x}$$

$\underline{f_{yx}(0,0) = 0}$, $\underline{f_{yx}(2,0) = 0}$

$D(0,0) = 4$, $f_{xx}(0,0) > 0$, so $\boxed{\text{local min. at } (0,0)}$

$D(2,0) = -4e^{-2} < 0$, so $\boxed{\text{saddle point at } (2,0)}$

$(D = f_{xx}f_{yy} - (f_{yx})^2)$

4(a) Find the maximum rate of change of $f(x, y) = x^2y + \sqrt{y}$ at the point $(2, 1)$. In which direction does it occur?

max. rate of change is $|\nabla f|$.

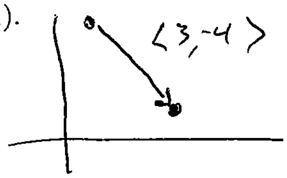
$$\left.\begin{array}{ll} f_x(x,y) = 2xy, & f_x(2,1) = 4 \\ f_y(x,y) = x^2 + \frac{1}{2\sqrt{y}}, & f_y(2,1) = 9/2 \end{array}\right\} \quad \nabla f(2,1) = \langle 4, 9/2 \rangle$$

$$|\nabla f| = \sqrt{16 + \frac{81}{4}} = \boxed{\frac{\sqrt{145}}{2}}$$

direction = direction of gradient $= \boxed{\langle 4, 9/2 \rangle}$

(or $\frac{2}{\sqrt{145}}\langle 4, 9/2 \rangle$, a unit vector)

(b) Find the directional derivative of $g(x, y) = 2\sqrt{x} - y^2$ at the point $(1, 5)$ in the direction toward the point $(4, 1)$.



$$\vec{u} = \frac{\langle 3, -4 \rangle}{|\langle 3, -4 \rangle|} = \underline{\langle 3/5, -4/5 \rangle}.$$

$$\nabla g(x,y) = \left\langle \frac{1}{\sqrt{x}}, -2y \right\rangle$$

$$\nabla g(1,5) = \langle 1, -10 \rangle.$$

$$D_{\vec{u}}\, g(1,5) = \langle 1, -10 \rangle \cdot \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle$$

$$= \frac{3}{5} + \frac{40}{5} = \boxed{\frac{43}{5}}$$

(c) Find du if $u = \ln(1 + se^{2t})$.

$$du = \frac{\partial u}{\partial s}\, ds + \frac{\partial u}{\partial t}\, dt$$

$$\boxed{du = \frac{1}{1+se^{2t}}(e^{2t})\, ds + \frac{1}{1+se^{2t}}(2se^{2t})\, dt}$$