1. Consider the function $f(x, y) = x^3 + y^3 - 3x - 3y$.

(a) Find all critical points.

(b) Use the Second Derivative Test to determine the behavior at each of these points.

(a) $f_x(x,y) = 3x^2 - 3 = 3(x^2 - 1)$

$f_y(x,y) = 3y^2 - 3 = 3(y^2 - 1)$

set both $= 0$: $x = \pm 1$, $y = \pm 1$

crit. points : $\boxed{(1,1),\ (1,-1),\ (-1,1),\ (-1,-1)}$

(b) $f_{xx}(x,y) = 6x$

$f_{xy}(x,y) = 0$ $\Rightarrow D = f_{xx} f_{yy} - (f_{xy})^2 = 36xy$

$f_{yy}(x,y) = 6y$

$(1,1)$: $D > 0$, $f_{xx} > 0$ $\Rightarrow$ local min at $(1,1)$

$(1,-1)$: $D < 0$ $\Rightarrow$ saddle point at $(1,-1)$

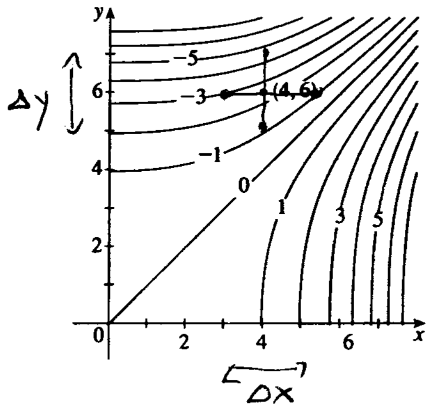
$(-1,1)$: $D < 0$ $\Rightarrow$ saddle point at $(-1,1)$

$(-1,-1)$: $D > 0$, $f_{xx} < 0$ $\Rightarrow$ local max at $(-1,-1)$

2(a) Level curves for a function $f(x, y)$ are shown below.

(i) Estimate $f_x(4, 6)$ and $f_y(4, 6)$.

(ii) Is $f_{xx}(4, 6)$ positive, negative, or zero? Explain briefly.



taking $\Delta x \approx 2$, get $\Delta f = (-1) - (-3) \approx 2$

so $\boxed{f_x(4,6) \approx \frac{2}{2} = 1}$

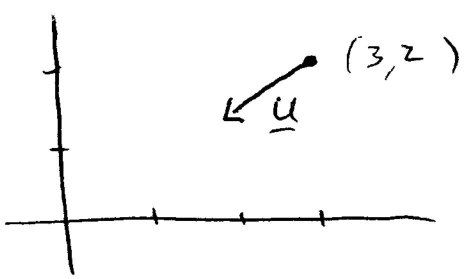
taking $\Delta y \approx 2$, get $\Delta f \approx (-5) - (-1) = 4$

so $\boxed{f_y(4,6) \approx \frac{-4}{2} = -2}$

Moving left to right at (4,6), horizontal spacing gets closer together, so f_x is increasing.

$\boxed{\text{So } f_{xx}(4,6) > 0}$.

(b) Find the directional derivative of $f(x, y) = x^2 + 4y^2$ at $(3, 2)$ in the direction of the origin. [Draw a picture!]



direction to the origin:

$(0,0) - (3,2) = \langle -3, -2 \rangle$

normalize:

$\underline{u} = \dfrac{\langle -3, -2 \rangle}{|\langle -3, -2 \rangle|} = \left\langle \dfrac{-3}{\sqrt{13}}, \dfrac{-2}{\sqrt{13}} \right\rangle$

$\nabla f(x,y) = \langle 2x, 8y \rangle$

$\nabla f(3,2) = \langle 6, 16 \rangle$

$D_{\underline{u}} f(3,2) = \underline{u} \cdot \nabla f(3,2) = \dfrac{-18}{\sqrt{13}} - \dfrac{32}{\sqrt{13}} = \boxed{\dfrac{-50}{\sqrt{13}}}$

3. Let $f(x,y) = x\sqrt{y}$. Use linear approximation (or a tangent plane) at an appropriate point (a,b) to estimate $f(0.99, 4.004)$:

- What is the appropriate point (a,b)?
- What is the linear approximation or tangent plane (your choice)?
- Estimate $f(0.99, 4.004)$.

$\boxed{\text{take } (a,b) = (1,4),}$

$f(1,4) = 2.$

$f_x(x,y) = \sqrt{y}, \quad f_x(1,4) = 2.$

$f_y(x,y) = \dfrac{x}{2\sqrt{y}}, \quad f_y(1,4) = \dfrac{1}{4}.$

Linear approximation at $(1,4)$ is:

$$\boxed{L(x,y) = 2(x-1) + \tfrac{1}{4}(y-4) + 2}$$

(or tangent plane is given by $z = 2(x-1) + \frac{1}{4}(y-4) + 2$)

Estimating:

$$f(0.99, 4.004) \approx L(0.99, 4.004)$$

$$= 2(0.99-1) + \tfrac{1}{4}(4.004-4) + 2$$

$$= 2(-0.01) + \tfrac{1}{4}(0.004) + 2$$

$$= -.02 + .001 + 2$$

$$= \boxed{1.981}$$

4. Let $\mathbf{r}(t) = \langle \sin 3t, t, \cos 3t \rangle$.
(a) Calculate the length of the curve for $0 \le t \le 2\pi$.
(b) Find the tangent vector $\mathbf{T}(t)$.
(c) Find the normal vector $\mathbf{N}(t)$.

(a) $\underline{r}'(t) = \langle 3\cos 3t, 1, -3\sin 3t \rangle$

$$|\underline{r}'(t)| = \sqrt{9\cos^2 3t + 1 + 9\sin^2 3t}$$

$$= \sqrt{10}$$

$$\text{length} = \int_0^{2\pi} |\underline{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{10}\,dt = \boxed{2\pi\sqrt{10}}$$

(b) $$\underline{T}(t) = \frac{\underline{r}'(t)}{|\underline{r}'(t)|} = \boxed{\left\langle \frac{3}{\sqrt{10}}\cos 3t, \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\sin 3t \right\rangle}$$

(c) $$\underline{N}(t) = \frac{\underline{T}'(t)}{|\underline{T}'(t)|}$$

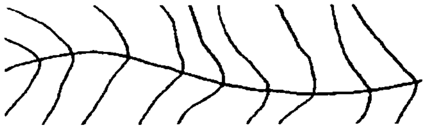
$$\underline{T}'(t) = \left\langle \frac{-9}{\sqrt{10}}\sin 3t, 0, \frac{-9}{\sqrt{10}}\cos 3t \right\rangle$$

$$|\underline{T}'(t)| = \sqrt{\left(\frac{-9}{\sqrt{10}}\right)^2 \sin^2 3t + \left(\frac{-9}{\sqrt{10}}\right)^2 \cos^2 3t}$$

$$= \frac{9}{\sqrt{10}}$$

$$\underline{N}(t) = \boxed{\langle -\sin 3t, 0, -\cos 3t \rangle}$$

Bonus (+2 points) Here is a map showing a mountain stream and some contour lines. Which way is the stream flowing, and why?



The question is, which contour lines are higher, which are lower.

If left is high, the stream is running along the top of a ridge (unlikely).

If right is high, the stream is running along the bottom of a valley (likely).

So flow is right to left.