2(a) Suppose we want to find the point on the surface $xy^2z^3 = 2$ that is closest to the origin. Write down a system of equations whose solution will include this closest point. [Do not solve the system or proceed any further.]

distance $= \sqrt{x^2+y^2+z^2}$

Minimize $f(x,y,z) = \text{dist}^2 = x^2+y^2+z^2$

subject to $g(x,y,z) = xy^2z^3 = 2$

Lagrange Multipliers: $\nabla f = \lambda \nabla g$:

$$\boxed{\begin{aligned} 2x &= \lambda y^2 z^3 \\ 2y &= \lambda 2xyz^3 \\ 2z &= \lambda 3xy^2z^2 \\ xy^2z^3 &= 2 \end{aligned}}$$

2(b) Find all critical points of the function $f(x, y) = 4xy^2 - x^2y^2 - xy^3$. Then, what can you say about the location of the absolute maximum of $f(x, y)$ on the rectangle $[0, 4] \times [0, 2]$?

$f_x = 4y^2 - 2xy^2 - y^3 = y^2(4-2x-y) = 0$ ①

$f_y = 8xy - 2x^2y - 3xy^2 = xy(8-2x-3y) = 0$ ②

By ①, $y=0$ or $4-2x-y=0$

$y = 4-2x$

↳ If $y=0$ then ② is already satisfied, so any $(x,0)$ is a C.P.

Say $y \neq 0$. Then $y = 4-2x$. ② ⇒ $x=0$ or $8-2x-3y=0$.

$x=0 \Rightarrow y=4$ so $(0,4)$.

$8-2x-3y=0$ ⇓ $x=1, y=2$ so $(1,2)$.

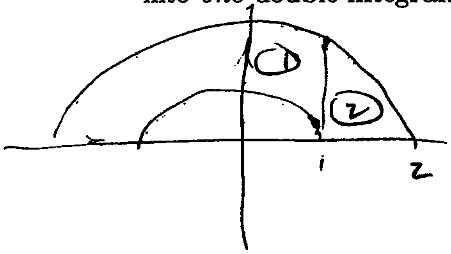
$\boxed{\text{CPs } (x,0), (0,4), (1,2)}$

$\boxed{\text{The abs. max is at a C.P. or is on the boundary.}}$

3(a) Use polar coordinates to combine the sum

$$\underbrace{\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}}_{1} xy\,dy\,dx \;+\; \underbrace{\int_1^2 \int_0^{\sqrt{4-x^2}}}_{2} xy\,dy\,dx$$

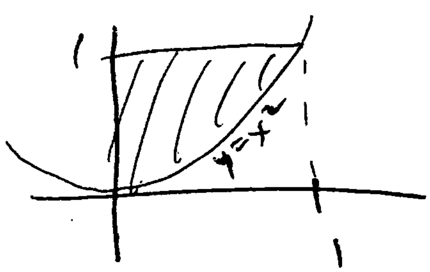
into *one* double integral. Do not evaluate the integral. [Draw the region!]



In polar coords:

$$\boxed{\int_0^{\pi/2} \int_1^2 r\cos\theta \, r\sin\theta \, r \, dr \, d\theta}$$

3(b) Sketch the region, and evaluate the integral $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\,dy\,dx$ by reversing the order of integration.



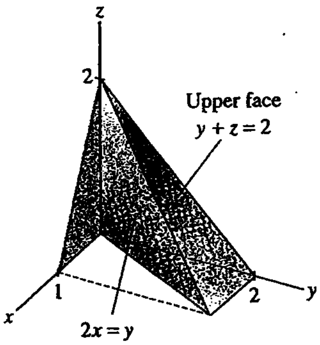
$$\int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\,dx\,dy$$

$$= \int_0^1 \frac{1}{4} x^4 \sin(y^3) \Big|_0^{\sqrt{y}} dy = \int_0^1 \frac{1}{4} y^2 \sin(y^3)\,dy$$

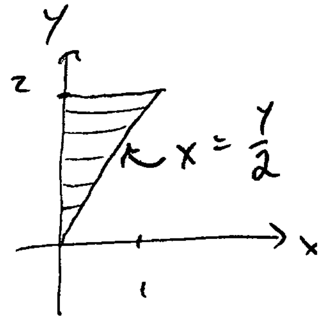
$$= \int_0^1 \frac{1}{12} \sin(u)\,du \qquad \left[\begin{array}{l} u = y^3 \\ du = 3y^2\,dy \end{array}\right]$$

$$= \frac{-1}{12} \cos(u) \Big|_0^1 = \boxed{\frac{-1}{12}(\cos(1) - 1)}$$

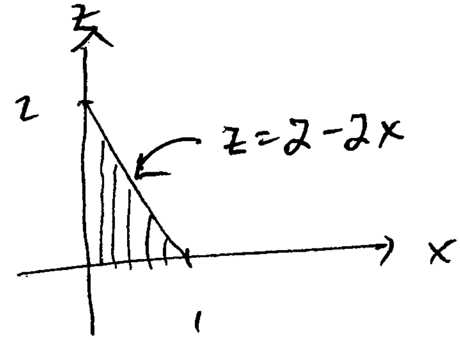
3. The figure below shows the region E bounded by $y+z=2$, $2x=y$, $x=0$, and $z=0$. Express $\iiint_E xe^z\,dV$ in three ways, using $dV = dzdxdy$, $dV = dydzdx$, and $dV = dxdydz$.



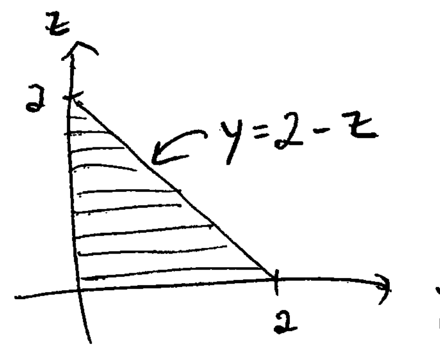
$$\boxed{\int_0^2\int_0^{\frac{y}{2}}\int_0^{2-y} xe^z\,dz\,dx\,dy}$$



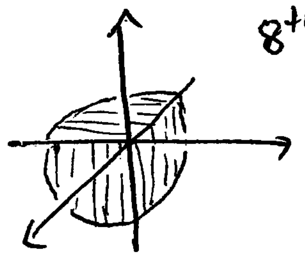
$$\boxed{\int_0^1\int_0^{2-2x}\int_{2x}^{2-z} xe^z\,dy\,dz\,dx}$$



$$\boxed{\int_0^2\int_0^{2-z}\int_0^{\frac{y}{2}} xe^z\,dx\,dy\,dz}$$



4(a) Express the integral $\iiint_E y\,dV$ in spherical coordinates, where E is the region $x^2+y^2+z^2 \le 1$, $x, y, z \le 0$.



8th octant

$y = \rho \sin\varphi \sin\theta$

$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$

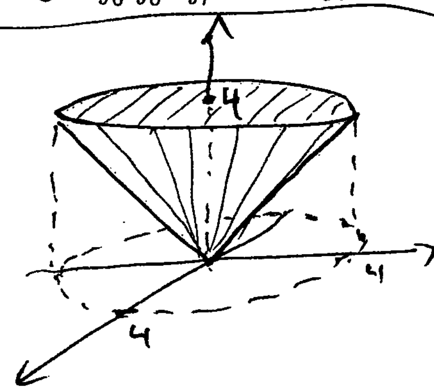
$$\boxed{\int_{\frac{\pi}{2}}^{\pi} \int_{\pi}^{\frac{3\pi}{2}} \int_0^1 (\rho \sin\varphi \sin\theta)\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi}$$

4(b) Sketch the solid whose volume is given by the integral $\int_0^4 \int_0^{2\pi} \int_r^4 r\,dz\,d\theta\,dr$, and evaluate this integral.

$r \le z \le 4$

$z = r$, cone

$z = 4$, plane



$$\int_0^4 \int_0^{2\pi} rz\Big|_r^4 \, d\theta \, dr = \int_0^4 \int_0^{2\pi} 4r - r^2 \, d\theta \, dr$$

$$= \int_0^{2\pi} d\theta \int_0^4 (4r - r^2)\,dr = 2\pi\left(2r^2 - \frac{r^3}{3}\Big|_0^4\right)$$

$$= 2\pi\left(32 - \frac{64}{3}\right) = \boxed{\frac{64\pi}{3}}$$