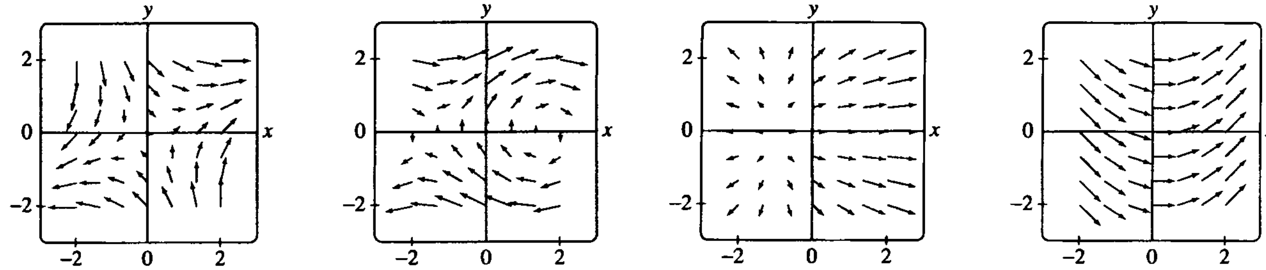
1(a) Match the vector fields with their plots below:

(i) $\mathbf{F}(x,y) = \langle 2, x\rangle$ D

(ii) $\mathbf{F}(x,y) = \langle 2x+2, y\rangle$ C

(iii) $\mathbf{F}(x,y) = \langle y, \cos x\rangle$ B

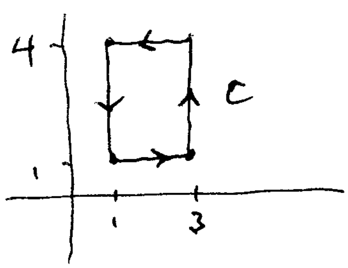
(iv) $\mathbf{F}(x,y) = \langle x+y, x-y\rangle$ A



(A) (B) (C) (D)

IV III II I

(b) Use Green's Theorem to evaluate the integral $\int_C (\ln x + y)\,dx - x^2\,dy$ where C is the rectangle with vertices $(1,1)$, $(3,1)$, $(1,4)$, and $(3,4)$.



$P: \ln x + y \qquad \frac{\partial P}{\partial y} = 1$

$Q: -x^2 \qquad \frac{\partial Q}{\partial x} = -2x$

$$\int_C (\ln x + y)\,dx - x^2\,dy \overset{G.T.}{=} \int_1^4\int_1^3 (-2x-1)\,dx\,dy$$

$$= \int_1^4 \left[-x^2 - x\right]_1^3 dy$$

$$= \int_1^4 (-9-3+1+1)\,dy$$

$$= 3(-10) = \boxed{-30}$$

2(a) Suppose one wants to find the points on the surface $x^2y + y^2z = 4$ that are closest to the origin.

(i) Write down the appropriate function to minimize.

(ii) Write down a system of equations that can be used to find the desired points. (Do not solve the system or proceed any further.)

distance from (x,y,z) to $(0,0,0)$ is $d = \sqrt{x^2+y^2+z^2}$.

(i) minimize $\boxed{f(x,y,z) = x^2+y^2+z^2}$ $(= d^2)$.

(ii) Lagrange multipliers: constraint is $g(x,y,z) = 4$,

$g(x,y,z) = x^2y + y^2z$.

$\nabla f = \langle 2x, 2y, 2z \rangle$, $\nabla g = \langle 2xy, x^2+2yz, y^2 \rangle$

system:

① $2x = \lambda \cdot 2xy$
② $2y = \lambda(x^2+2yz)$
③ $2z = \lambda \cdot y^2$
④ $x^2y + y^2z = 4$

2(b) Find the cosine of the angle between the planes $x + y - z = 1$ and $2x - 3y + 4z = 5$. Explain your reasoning.

The angle between the planes equals the angle between their normal vectors.

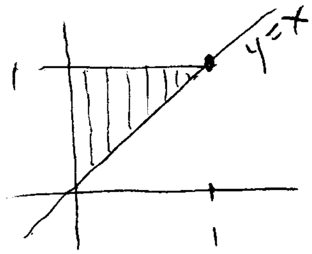
These vectors are $\underline{a} = \langle 1, 1, -1 \rangle$ and $\underline{b} = \langle 2, -3, 4 \rangle$.

$$\cos\theta = \frac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|} = \frac{2-3-4}{\sqrt{1^2+1^2+1^2}\,\sqrt{2^2+3^2+4^2}}$$

$$= \boxed{\frac{-5}{\sqrt{87}}}$$

3(a) Reverse the order of integration for the integral $\int_0^1 \int_x^1 \tan(y^2)\, dy\, dx$. (Do not solve the integral.)

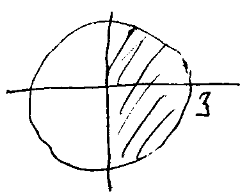
region:



going horizontally first, we get

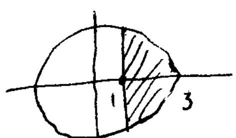
$$\boxed{\int_0^1 \int_0^y \tan(y^2)\, dx\, dy}$$

3(b) Express the integral $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\, dy\, dx$ in polar coordinates. (Do not solve the integral.)



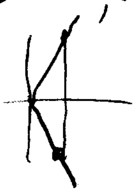
$$\boxed{\int_{-\pi/2}^{\pi/2} \int_0^3 (r^3\cos^3\theta + r^3\cos\theta\sin^2\theta)\, r\, dr\, d\theta}$$

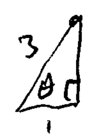
If



inner curve is $x=1$, ie $r\cos\theta = 1$, or $\underline{r = \frac{1}{\cos\theta}}$

also need endpoints for θ:

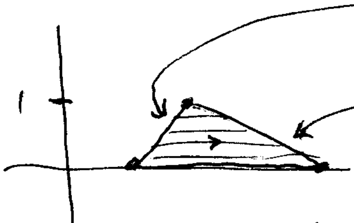




$\cos\theta = \frac{1}{3}$

get $\boxed{\int_{-\cos^{-1}\frac{1}{3}}^{\cos^{-1}\frac{1}{3}} \int_{\frac{1}{\cos\theta}}^{3} (r^3\cos^3\theta + r^3\cos\theta\sin^2\theta)\, r\, dr\, d\theta}$

3(c) Express the volume of E as an iterated integral, where E is the region under the surface $z = x^2y$ and above the triangle in the xy-plane with vertices $(1,0)$, $(2,1)$, and $(4,0)$. (Do not solve the integral.)



$y = x - 1 \rightsquigarrow x = y + 1$

$y = 2 - \frac{1}{2}x \rightsquigarrow x = 4 - 2y$

$$\text{vol}(E) = \boxed{\int_0^1 \int_{y+1}^{4-2y} x^2y\, dx\, dy} \quad \text{OR} \quad \boxed{= \int_0^1 \int_{y+1}^{4-2y} \int_0^{x^2y} dz\, dx\, dy}$$

4. Evaluate the line integrals:

(a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle y, z, x \rangle$ and C is given by $\mathbf{r}(t) = \langle \sqrt{t}, t, t^2 \rangle$, $1 \le t \le 4$.

$$\underline{r}'(t) = \left\langle \frac{1}{2\sqrt{t}}, 1, 2t \right\rangle$$

$$\underline{F}(\underline{r}(t)) = \langle t, t^2, \sqrt{t} \rangle$$

$$\int_1^4 \langle t, t^2, \sqrt{t} \rangle \cdot \left\langle \frac{1}{2\sqrt{t}}, 1, 2t \right\rangle dt = \int_1^4 \frac{1}{2}\sqrt{t} + t^2 + 2t^{3/2}\, dt$$

$$= \frac{1}{3}t^{3/2} + \frac{1}{3}t^3 + \frac{4}{5}t^{5/2} \Big|_1^4$$

$$= \boxed{\frac{1}{3}(8) + \frac{1}{3}(64) + \frac{4}{5}(32) - \frac{1}{3} - \frac{1}{3} - \frac{4}{5}}$$

(b) $\int_C xe^{yz}\, ds$ where C is the curve given by $\mathbf{r}(t) = \langle t, 2t, 3t \rangle$, $0 \le t \le 1$.

$$\underline{r}'(t) = \langle 1, 2, 3 \rangle$$

$$ds = \sqrt{1^2 + 2^2 + 3^2}\, dt = \sqrt{14}\, dt$$

$$\int_0^1 t e^{6t^2} \sqrt{14}\, dt \qquad u = 6t^2;\ du = 12t\, dt$$

$$= \frac{\sqrt{14}}{12} \int_0^6 e^u\, du = \frac{\sqrt{14}}{12} \left[e^u \right]_0^6$$

$$= \boxed{\frac{\sqrt{14}}{12}(e^6 - 1)}$$

5(a) Show that the vector field $\mathbf{F}(x,y,z) = (2x\sin y)\mathbf{i} + (x^2\cos y + z)\mathbf{j} + y\mathbf{k}$ is conservative and use this fact to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the line segment from $(1,0,1)$ to $(2,\pi/2,0)$.

Want f such that $\nabla f = \langle 2x\sin y,\ x^2\cos y + z,\ y\rangle$.

$f_x = 2x\sin y \Rightarrow f = x^2\sin y + C(y,z)$

$f_y = x^2\cos y + z \Rightarrow f = x^2\sin y + yz + D(x,z)$

$f_z = y \Rightarrow f = yz + E(x,y)$

$\underline{f = x^2\sin y + yz}$ works.

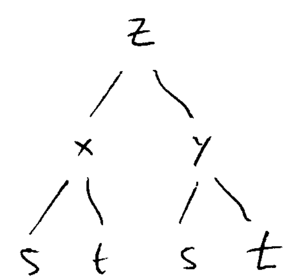
FTLI: $\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,\frac{\pi}{2},0) - f(1,0,1)$

$= \boxed{4}$

5(b) Suppose $z = f(x,y)$ and $x = g(s,t)$, $y = h(s,t)$, where $g(1,2) = 3$, $g_s(1,2) = -1$, $g_t(1,2) = 4$, $h(1,2) = 6$, $h_s(1,2) = -5$, $h_t(1,2) = 10$, $f_x(3,6) = 7$, $f_y(3,6) = 8$.

(i) Write down the chain rule for $\dfrac{\partial z}{\partial s}$ and for $\dfrac{\partial z}{\partial t}$.

(ii) Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $s = 1$ and $t = 2$.



$$\boxed{\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}\\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\end{aligned}}$$

$s=1, t=2 \Rightarrow x = g(1,2) = 3$

$y = h(1,2) = 6$

$\frac{\partial z}{\partial y} = f_y(3,6) = 8 \qquad \frac{\partial z}{\partial x} = f_x(3,6) = 7$

$\frac{\partial y}{\partial s} = h_s(1,2) = -5 \qquad \frac{\partial y}{\partial t} = h_t(1,2) = 10$

$\frac{\partial x}{\partial s} = g_s(1,2) = -1 \qquad \frac{\partial x}{\partial t} = g_t(1,2) = 4$

$\frac{\partial z}{\partial s} = (8)(-5) + (7)(-1)$

$= \boxed{-47}$

$\frac{\partial z}{\partial t} = (8)(10) + (7)(4)$

$= \boxed{108}$