1(a) Find

(i) 13 **mod** 3 and 13 **div** 3

(ii) -4 **mod** 3 and -4 **div** 3

(b) Suppose a, b, c, m, n are integers with $n, m > 1$. Show that if $n|m$ and $a \equiv b$ (mod m), then $a \equiv b$ (mod n).

(a) $13 = 4 \cdot 3 + 1$

so $\boxed{\begin{array}{l} 13 \text{ mod } 3 = 1 \\ 13 \text{ div } 3 = 4 \end{array}}$

$-4 = (-2) \cdot 3 + 2$

so $\boxed{\begin{array}{l} -4 \text{ mod } 3 = 2 \\ -4 \text{ div } 3 = -2 \end{array}}$

(b) $a \equiv b \pmod m$ implies that $m \mid (a-b)$.

Since $n|m$ and $m \mid (a-b)$

we have $n \mid (a-b)$

(write $m = kn$, $(a-b) = lm$; then $(a-b) = lkn$)

hence $a \equiv b \pmod n$.

2(a) For each positive integer i let $A_i = (0, i) = \{x \in \mathbb{R} \mid 0 < x < i\}$. Find the sets $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

(b) List all the elements of the set $P(\{a, b, \{a, b\}\})$, where $a \neq b$.

(a) $\bigcup_{i=1}^{\infty} A_i = \{x \in \mathbb{R} \mid x \in A_i \text{ for some } i\}$

$= \boxed{(0, \infty)}$

$\bigcap_{i=1}^{\infty} A_i = \{x \in \mathbb{R} \mid x \in A_i \text{ for all } i\}$

$= \boxed{(0, 1)}$

(b)

The elements of $P(\{a, b, \{a,b\}\})$ are:

$\emptyset$, $\{a\}$, $\{b\}$, $\{\{a,b\}\}$,

$\{a, b\}$, $\{a, \{a,b\}\}$, $\{b, \{a,b\}\}$,

$\{a, b, \{a,b\}\}$.

3(a) Find gcd(21, 44).

(b) Find integers s and t such that $21s + 44t = \gcd(21, 44)$.

$(44, 21)$

(1) $44 = 21 \cdot 2 + 2$ $\qquad (21, 2)$

(2) $21 = 2 \cdot 10 + 1$ $\qquad (2, 1)$

(3) $2 = 1 \cdot 2 + 0$ $\qquad (1, 0)$

$\gcd(44, 21) = \gcd(21, 2) = \gcd(2, 1) = \boxed{1}.$

By (2), $1 = 21 + (-10) \cdot 2$

by (1), $2 = 44 + (-2) \cdot 21$

so $1 = 21 + (-10)(44 + (-2) \cdot 21)$

$1 = (-10)44 + (21) \cdot 21$

$\boxed{s = 21, \ t = -10 \text{ work.}}$

4(a) Prove or disprove: for all $x, y \in R$, $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$.

(b) Prove or disprove: for all $x \in \mathbb{R}$, $\lceil x \rceil = \lfloor x \rfloor + 1$.

(c) For the functions $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ below, determine whether they are onto (explain!):
(i) $f(m,n) = m - n$
(ii) $f(m,n) = m^2 + n^2$

(a) it's false: take $x = y = \frac{1}{2}$.

Then $\lceil x+y \rceil = \lceil 1 \rceil = 1$

$\lceil x \rceil + \lceil y \rceil = 1+1 = 2$.

(b) it's false: take any integer n.

$\lceil n \rceil = n$

$\lfloor n \rfloor + 1 = n + 1$.

(c)(i) for any $k \in \mathbb{Z}$, $f(k,0) = k$
so f is onto.

(ii) $f(m,n) \geq 0$ for any m, n
so f is not onto.
(-1, for example, is not in the image)