1(a) Consider the relations R_1, R_2 on real numbers defined by

$$R_1 = \{(a,b) \mid a < b\} \quad \text{and} \quad R_2 = \{(a,b) \mid a + b \leq 3\}.$$

Which of these are antisymmetric? Which are transitive? Say briefly why or why not in each case.

R_1 is antisymmetric: aR_1b and bR_1a never occurs

R_2 is not antisymmetric: eg. $1R_22$ and $2R_21$ hold

R_1 is transitive: if $a<b$ and $b<c$ then $a<c$.

R_2 is not transitive: eg. $2R_21$ and $1R_22$ but $2 \not R_2 2$.

(b) Let d be a positive integer. Show that among any collection of $d+1$ distinct integers, there are two with the same remainder when they are divided by d.

There are d possible remainders: $\{0, 1, 2, \cdots, d-1\}$.

If I have integers $n_1, n_2, \cdots, n_{d+1}$ which are all distinct, assign each its remainder.

Since there are more integers than possible remainders, the Pigeonhole Principle says that some remainder appears at least twice.

2(a) Define what it means for a set A to be *countable*. Then, prove that if A and B are countable, infinite sets with $A \cap B = \emptyset$, then $A \cup B$ is countable.

- A is countable if it is finite or there is a bijection $f: \mathbb{N} \to A$ (or $A \to \mathbb{N}$).
- Given A and B, there are bijections $f_A: \mathbb{N} \to A$ and $f_B: \mathbb{N} \to B$. Define $g: \mathbb{N} \to A \cup B$ by

$$g(n) = \begin{cases} f_A(n/2) & \text{if } n \text{ is even} \\ f_B((n+1)/2) & \text{if } n \text{ is odd.} \end{cases}$$

- g is 1-1 on even numbers (since f_A is 1-1) and on odd numbers (since f_B is 1-1) and g never maps an even number and an odd number to the same element of $A \cup B$ (since A and B are disjoint). Hence g is 1-1.
- Given $x \in A \cup B$, if $x \in A$ then $x = f_A(k)$ for some k. Then $x = g(2k)$. If $x \in B$ then $x = f_B(k)$ for some k, and $x = g(2k-1)$. Hence g is onto.

(b) Write out the expansion of $(x + 2y)^6$, using binomial coefficients.

$$(x+2y)^6 = \sum_{j=0}^{6} \binom{6}{j} x^{6-j} (2y)^j =$$

$$\binom{6}{0}x^6 + \binom{6}{1}x^5(2y) + \binom{6}{2}x^4(2y)^2 + \binom{6}{3}x^3(2y)^3 +$$

$$\binom{6}{4}x^2(2y)^4 + \binom{6}{5}x(2y)^5 + \binom{6}{6}(2y)^6$$

3. Using induction, prove that $n^2 \geq 2n+1$ when n is an integer greater than or equal to 3. What is the base case? What is the induction hypothesis?

Base: $n=3$. $n^2 = 9$, $2n+1 = 7$, $9 \geq 7$ ✓

Ind. Step: Assume $n^2 \geq 2n+1$, Prove $(n+1)^2 \geq \underbrace{2(n+1)+1}_{=2n+3}$:

$(n+1)^2 = n^2 + 2n + 1 \geq 2n+1 + 2n+1$ by ind. hyp.

$= 4n+2$.

Is $4n+2 \geq 2n+3$?

$\Leftrightarrow 2n \geq 1$

$\Leftrightarrow n \geq \frac{1}{2}$, which is true.

So: $(n+1)^2 \geq 4n+2 \geq 2n+3 = 2(n+1)+1$. ✓

Bonus. Prove that $2^n > n^2$ when n is an integer greater than or equal to 5.

Base: $n=5$: $2^5 = 32$, $5^2 = 25$, $32 > 25$ ✓

Ind. Step. Assume $2^n > n^2$, Prove $2^{n+1} > (n+1)^2$:

$2^{n+1} = 2 \cdot 2^n = 2^n + 2^n > n^2 + n^2$ by ind hyp.

$\geq n^2 + (2n+1)$ by part (a)

$= (n+1)^2$. ✓

4(a) A group comtains n women and n men. How many ways are there to arrange these people in a row if the men and women alternate? Explain.

2 patterns: (1) MWMW --- MW and (2) WMWM --- WM

For (1), $n!$ ways to arrange men, $n!$ ways to arrange women $\Rightarrow (n!)(n!)$ ways.

For (2) similar, $(n!)(n!)$ ways.

Sum rule: $(n!)(n!) + (n!)(n!) = \boxed{2(n!)^2 \text{ ways.}}$

(b) A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes contain at most three tails? (You do not need to evaluate your expression completely.)

four non-overlapping cases: 0 tails, 1 tail, 2 tails, 3 tails. Sum rule gives

$$\boxed{\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}}$$

possible outcomes.

(if k tails, choose k out of 10 positions where tails occur, eg.

HH T H T HH T HH
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