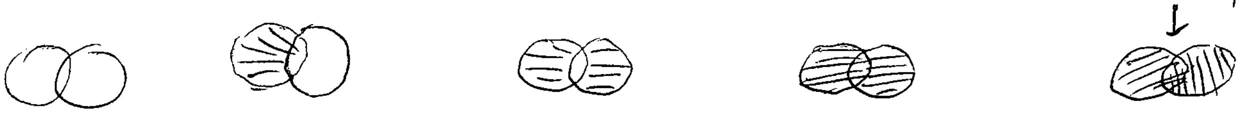
1(a) Let A and B be subsets of a finite set U. List the following in order of increasing size: $|A-B|$, $|A \oplus B|$, $|A|+|B|$, $|A \cup B|$, $|\emptyset|$.

overlap counted twice



$$|\emptyset| \leq |A-B| \leq |A \oplus B| \leq |A \cup B| \leq |A|+|B|$$

(b) Let $f\colon \{1,2,3,4\} \to \{a,b,c,d\}$ and $g\colon \{a,b,c,d\} \to \{1,2,3,4\}$ be functions given by $f(1)=d$, $f(2)=c$, $f(3)=a$, $f(4)=b$ and $g(a)=2$, $g(b)=1$, $g(c)=3$, $g(d)=2$.

(i) Is f one-to-one? Is g one-to-one?

(ii) Is f onto? Is g onto?

(iii) Does either f or g have an inverse? If so, find this inverse.

(i) f is one-to-one (no two numbers are sent to the same letter)

g is not one-to-one since $g(a) = g(d)$

(ii) f is onto (every letter is f(something))

g is not onto: 4 is not in the image

(iii) only f has an inverse:

$f^{-1}(a) = 3$

$f^{-1}(b) = 4$

$f^{-1}(c) = 2$

$f^{-1}(d) = 1$

2. Use the Euclidean Algorithm to find gcd(78, 35). Then find integers s nad t such that $78s + 35t = \gcd(78, 35)$.

① $78 = 2 \cdot 35 + 8$

② $35 = 4 \cdot 8 + 3$

③ $8 = 2 \cdot 3 + 2$

④ $3 = 1 \cdot 2 + 1$ ← $\boxed{\gcd(78,35) = 1}$

$2 = 2 \cdot 1 + 0$

④ gives: $1 = (1)\cdot 3 + (-1)\cdot 2$

③ gives $2 = (1)\cdot 8 + (-2)\cdot 3$

Hence $1 = (1)3 + (-1)\big((1)8 + (-2)3\big)$

$= (-1)8 + (3)3$

② gives $3 = (1)35 + (-4)8$

Hence $1 = (-1)8 + (3)\big((1)35 + (-4)8\big)$

$= (3)35 + (-13)8$

① gives $8 = (1)78 + (-2)35$

Hence $1 = (3)35 + (-13)\big((1)78 + (-2)35\big)$

$1 = (-13)78 + (29)35$

$\boxed{s = -13, \quad t = 29}$

3. Consider the system of congruences

$$x \equiv 1 \pmod 4), \quad x \equiv 0 \pmod 3).$$

(i) What does the Chinese remainder theorem say about the set of solutions x to the system?
(ii) Find one solution (using any method – think about it!).
(iii) Describe all solutions in $\mathbb{Z}$.

(i) since 4 and 3 are rel. prime, there is a unique solution modulo 12

(ii) there will be a solution between 1 and 12. x is a multiple of 3, so possibilities are 3, 6, 9, 12. Of these, $\boxed{\text{9 works}}$ because 9−1 is a multiple of 4.

(iii) the solutions are all numbers of the form $\boxed{9 + 12n,\ n \in \mathbb{Z}.}$

4. Use induction to prove that if n people stand in line, and the person in the front of the line is a woman, and the person in the back of the line is a man, then somewhere in the line there is a woman standing directly in front of a man.

Let $P(n)$ = statement in box. Using induction on n:

Base Case is $n = 2$. Then the woman is already directly in front of the man, so the statement is true.

Induction Step Consider $n+1$:

$$M \underbrace{\qquad\qquad}_{n-1 \text{ people}} W$$

back $\longrightarrow$ front

There are two cases If the second person in line is a man, then the woman in front satisfies the conclusion.

Otherwise, the second person is a woman. In this case, if we omit the woman in the front, we have n people with a woman in front and a man in back. $(M \underbrace{\quad}_{n-2} W) W$

By induction hypothesis, there is a woman directly in front of a man among these n people.

5(a) Give the formulas for $P(n,r)$ and $C(n,r)$.

$$P(n,r) = \frac{n!}{(n-r)!} \qquad C(n,r) = \frac{n!}{(n-r)!\,r!}$$

(b) Give a combinatorial proof that if $1 \le k \le n$ then $k\binom{n}{k} = n\binom{n-1}{k-1}$. [Hint: show that each side of the identity counts the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]

One way of thinking: there are n players and we want to choose a team of k, and a team captain. How many ways?

Method 1: choose team ($\binom{n}{k}$ ways)
then choose captain (k ways)

Product rule: $\boxed{k\binom{n}{k} \text{ ways.}}$

Method 2: choose captain first, from all the players (n ways)
then choose the rest of the team from the remaining $n-1$ players ($\binom{n-1}{k-1}$ ways).

Product Rule: $\boxed{n\binom{n-1}{k-1} \text{ ways.}}$

$\boxed{\text{Hence } k\binom{n}{k} = n\binom{n-1}{k-1}}$, since these quantities count the same thing.

6(a) What properties must a relation satisfy to be an equivalence relation?

It must be reflexive, symmetric, and transitive.

(b) Let $\sim$ be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$(a, b) \sim (c, d) \text{ if } a + d = b + c.$$

Verify that this is an equivalence relation.

reflexive $(a,b) \sim (a,b)$ is true because $a+b = b+a$.

symmetric if $(a,b) \sim (c,d)$ then $a+d = b+c$

hence $c+b = d+a$, so $(c,d) \sim (a,b)$

transitive if $(a,b) \sim (c,d) \sim (e,f)$ then $a+d = b+c$ and $c+f = d+e$.

Adding eqns gives: $a + \cancel{d} + \cancel{c} + f = b + \cancel{c} + \cancel{d} + e$

so $a+f = b+e$

which means that $(a,b) \sim (e,f)$.

(c) What is the equivalence class of $(1, 2)$?

all (a,b) such that $(1,2) \sim (a,b)$

i.e. $1+b = a+2$

i.e. $b = a+1$

$$\boxed{[(1,2)] = \{ (a, a+1) \mid a \in \mathbb{Z} \}}$$