1. Show (using any method) that if A and B are sets then $(A \cap B) \cup (A \cap \overline{B}) = A$.

<u>Step 1</u> $(A \cap B) \cup (A \cap \overline{B}) \supset A$:

Suppose $x \in A$. Two cases:

- If $x \in B$ then $x \in A \cap B$ and so $x \in (A \cap B) \cup (A \cap \overline{B})$.
- If $x \notin B$ then $x \in \overline{B}$ so $x \in A \cap \overline{B}$ and hence $x \in (A \cap B) \cup (A \cap \overline{B})$.

So, in both cases, $x \in (A \cap B) \cup (A \cap \overline{B})$.

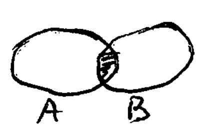
<u>Step 2</u> $(A \cap B) \cup (A \cap \overline{B}) \subset A$:

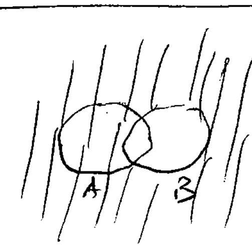
If $x \in (A \cap B) \cup (A \cap \overline{B})$ then either $x \in A \cap B$ or $x \in A \cap \overline{B}$.

- If $x \in A \cap B$ then $x \in A$.
- If $x \in A \cap \overline{B}$ then $x \in A$.

So, either way, $x \in A$. $\square$

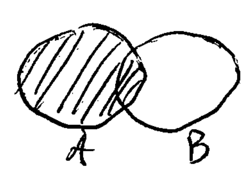
<u>Venn Diagrams</u> :

$A \cap B$: 

, $\overline{B}$: 

,

$A \cap \overline{B}$: 

, so $(A \cap B) \cup (A \cap \overline{B}) =$ 

— same as A.

2(a) Let a, b, c be integers. Prove that if $a|b$ and $b|c$ then $a|c$.

Since $a|b$, we have $b = ka$ for some $k \in \mathbb{Z}$.
Since $b|c$, we have $c = \ell b$ for some $\ell \in \mathbb{Z}$.

now $c = \ell b = (\ell k)a$
and hence $a|c$.

2(b) If $A = \{a, b\}$, write out the set (as a list of elements) $P(A) \times A$. (Recall that P denotes the power set.)

$P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

so

$P(A) \times A =$
$\{(\emptyset, a), (\emptyset, b), (\{a\}, a), (\{a\}, b),$
$(\{b\}, a), (\{b\}, b), (\{a,b\}, a), (\{a,b\}, b)\}$

3(a) Find the domain and image of the function that assigns to each positive integer the number of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ that appear as decimal digits in the integer.

Since the function is defined for each positive integer, the domain is $\mathbb{Z}_+ = \{1, 2, \dots\}$.

A positive integer involves at least one digit, and may use as many as all ten.

So the image is $\{1, 2, \dots, 10\}$.

3(b) Determine which of these functions $f\colon \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ is *onto* (and give brief explanations):

(i) $f(m, n) = 2m - n$

(ii) $f(m, n) = m^2 - 4$

(iii) $f(m, n) = |m| - |n|$

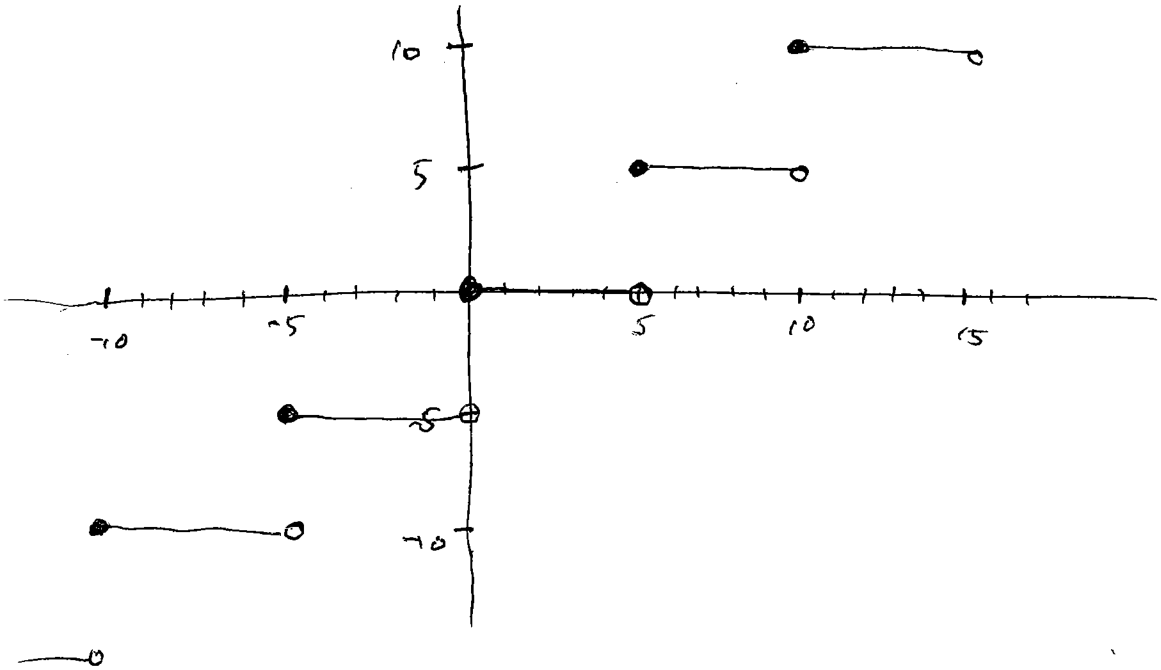
(i) onto, since for any $k \in \mathbb{Z}$, $f(k, k) = 2k - k = k$.

(ii) not onto, since $m^2 - 4$ is always -4 or greater.

(iii) onto: if $k \in \mathbb{Z}_+$ then $f(k, 0) = k$

if $k \in \mathbb{Z}_-$ then $f(0, -k) = k$.

4(a) Graph carefully the function $f(x) = 5\lfloor x/5 \rfloor$. What is the image of f?



4(b) Find $f^{-1}(\{x \mid 5 < x < 10\})$ and $f^{-1}(\{15, 20\})$.

$$f^{-1}(\{x \mid 5 < x < 10\}) = \{x \mid 5 < f(x) < 10\}$$
$$= \emptyset.$$

$$f^{-1}(\{15, 20\}) = \{x \mid f(x) \in \{15, 20\}\}$$
$$= \{x \mid f(x) = 15 \text{ or } f(x) = 20\}$$
$$= [15, 20) \cup [20, 25)$$
$$= [15, 25)$$