1. Use induction to prove that $n! < n^n$ for every integer $n \geq 2$:
What is the basis step?
What is the inductive hypothesis?
What do you need to prove in the inductive step?
Complete these steps, and the proof.

Let $P(n)$ be "$n! < n^n$".

basis step: $P(2)$, or "$2! < 2^2$"

this is true: $\underset{2}{\overset{\|}{2!}} < \underset{4}{\overset{\|}{2^2}}$.

inductive hypothesis: $P(k-1)$, or "$(k-1)! < (k-1)^{k-1}$"

need to prove: $P(k-1) \rightarrow P(k)$.

we have: $(k-1)! < (k-1)^{k-1}$, want to prove $k! < k^k$.

write $k! = k \cdot (k-1)!$.

Then $k \cdot (k-1)! < k \cdot (k-1)^{k-1}$ by ind. hyp.

$< k \cdot k^{k-1}$ since $k-1 < k$

$= k^k$.

So $k! < k^k$, i.e. $P(k)$ is true.

Since we have proved $P(2)$ and $P(k-1) \rightarrow P(k)$ for $k \geq 2$, by induction, we have proved $P(n)$ for all $n \geq 2$.

2(a) Use the Euclidean Algorithm to find gcd(9888, 6060).

$9888 = 1 \cdot \underline{6060} + \underline{3828}$

$6060 = 1 \cdot \underline{3828} + \underline{2232}$

$3828 = 1 \cdot \underline{2232} + \underline{1596}$

$2232 = 1 \cdot \underline{1596} + \underline{636}$

$1596 = 2 \cdot \underline{636} + \underline{324}$

$636 = 1 \cdot \underline{324} + \underline{312}$

$324 = 1 \cdot \underline{312} + \underline{12}$

$312 = 26 \cdot \underline{12} + \underline{0}$

So $\gcd(9888, 6060)$

$= \gcd(312, 12)$

$= \boxed{12}$

2(b) Find the greatest common divisor and least common multiple of the numbers $3^7 5^3 7^3$ and $2^{11} 3^5 5^9$.

$\boxed{\begin{array}{c} \text{gcd} \\ \| \\ 3^5 5^3 \end{array}}$ $(= 2^0 3^5 5^3 7^0)$ using min. of each exponent

$\boxed{\text{lcm} = 2^{11} 3^7 5^9 7^3}$ using max. of each exponent

2(c) Explain why $ab = gcd(a, b) \cdot lcm(a, b)$ for all positive integers a, b.

if $a = p_1^{m_1} \cdots p_k^{m_k}$, $b = p_1^{n_1} \cdots p_k^{n_k}$ with each p_i prime and each $m_i, n_i \geq 0$, then

$\underline{\gcd(a,b) = p_1^{\min(m_1, n_1)} \cdots p_k^{\min(m_k, n_k)}}$, $\underline{\text{lcm}(a,b) = p_1^{\max(m_1, n_1)} \cdots p_k^{\max(m_k, n_k)}}$

and $\underline{ab = p_1^{m_1+n_1} \cdots p_k^{m_k+n_k}}$. The equation holds

because $\max(m_i, n_i) + \min(m_i, n_i) = m_i + n_i$ for each i.

3(a) Give the definition for an infinite set S to be *countable*.

Then, suppose that S is countable and F is a finite set with n elements, disjoint from S. Prove that $F \cup S$ is countable.

S is countable if there is a bijection $S \xrightarrow{f} \mathbb{N}$.

Let $F = \{a_0, \dots, a_{n-1}\}$.

To show $F \cup S$ is countable, we define a function

$$F \cup S \xrightarrow{g} \mathbb{N} \text{ by: } g(x) = \begin{cases} i & \text{if } x = a_i \in F \\ f(x)+n & \text{if } x \in S \end{cases}.$$

(shift elts of S forward by n, map F to the freed-up numbers $0, \dots, n-1$.)

<u>g is onto</u>: every $i \in \mathbb{N}$ is either $g(a_i)$ (if $i < n$) or is $g(f^{-1}(i-n))$.

<u>g is 1-1</u>: g is clearly injective on F, and is injective on S since if $g(x) = g(y)$, then $f(x)+n = f(y)+n$, so $f(x) = f(y)$, hence $x = y$ (since f is 1-1). Finally, if $a_i \in F$ and $x \in S$, then $g(a_i) \neq g(x)$, since $g(a_i) < n$ and $g(x) \geq n$.

3(b) What is the coefficient of x^9 in $(2-x)^{19}$?

By the binomial theorem, $(2-x)^{19} = \sum_{k=0}^{19} \binom{19}{k} (2)^k (-x)^{19-k}$.

The x^9-term occurs when $k = 10$, so this term is

$$\binom{19}{10} 2^{10} (-x)^9 = \binom{19}{10} 2^{10} (-1)^9 x^9$$

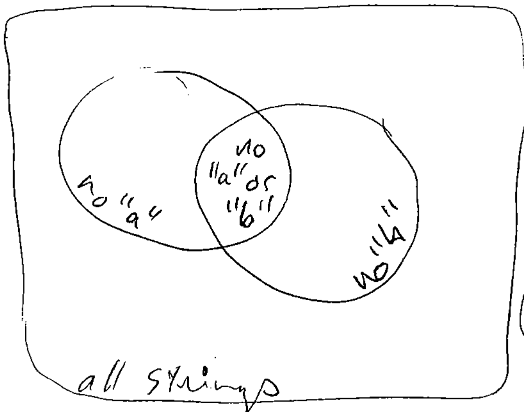
$$= \underbrace{\frac{-19!\,2^{10}}{9!\,10!}}\, x^9$$

This is the coefficient ↺.

4. How many strings of six lowercase letters from the English alphabet contain
the letter a?
the letters a and b?
the letters a and b in consecutive positions with a preceding b, with all the letters distinct?

(i) total number of strings $= 26^6$
strings with no "a" : 25^6
number of strings containing "a" : $\boxed{26^6 - 25^6}$

(ii)



$\#$ strings with no "a" : 25^6
$\#$ strings with no "b" : 25^6
$\#$ strings with no "a" or "b" : 24^6

By inclusion-exclusion, the union of these sets has cardinality
$25^6 + 25^6 - 24^6$.

So the number of strings containing "a" and "b" is $\boxed{26^6 - (25^6 + 25^6 - 24^6)}$.

(iii) there are 5 configurations:
$ab\underline{\quad}_4$, $\underline{\quad}_1 ab \underline{\quad}_3$, $\underline{\quad}_2 ab \underline{\quad}_2$, $\underline{\quad}_3 ab \underline{\quad}_1$, $\underline{\quad}_4 ab$.

In each configuration, the remaining letters form a 4-permutation of $\{c, d, \ldots, z\}$. So there are $24 \cdot 23 \cdot 22 \cdot 21$ possibilities for each.

In all, $\boxed{5 \cdot 24 \cdot 23 \cdot 22 \cdot 21}$ strings.