1. Let $f(x, y) = y^2 \cos(x) + 4(x^3 + 1)y$.

(a) At the point $(0, 2)$, find either the linearization $L(x, y)$ or an equation of the tangent plane (your choice).

(b) Estimate $f(0.02, 2.01)$. [A calculator should not be necessary.]

(a) $f_x(x,y) = -y^2 \sin x + 12x^2 y$

$f_y(x,y) = 2y \cos x + 4(x^3+1)$

$f_x(0,2) = 0$, $f_y(0,2) = 4 + 4 = 8$.

$f(0,2) = 4 + 8 = 12$.

$L(x,y) = 12 + 0(x-0) + 8(y-2)$

$\boxed{L(x,y) = 8y - 4}$

or

$\boxed{\text{tangent plane: } z = 8y - 4}$

(b) $f(0.02, 2.01) \approx L(0.02, 2.01)$

$= 8(2.01) - 4$

$= 16.08 - 4$

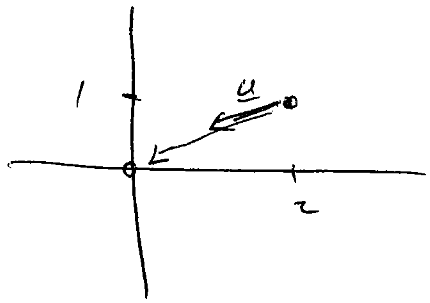
$= \boxed{12.08}$

2. Let $f(x,y) = x^3 - 4xy$.

(a) Find the directional derivative of $f(x,y)$ at $(2,1)$ in the direction of the origin.

(b) Find the maximum rate of change (among all directions) of $f(x,y)$ at $(2,1)$, and the direction in which it occurs.

(a)



$$\underline{u} = \frac{\langle -2,-1\rangle}{|\langle -2,-1\rangle|} = \left\langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle.$$

$$f_x(x,y) = 3x^2 - 4y, \quad f_y(x,y) = -4x$$

$$\nabla f = \langle 3x^2 - 4y, -4x\rangle$$

$$\nabla f(2,1) = \langle 8, -8\rangle$$

Directional derivative $= \nabla f(2,1) \cdot \underline{u}$

$$= \langle 8,-8\rangle \cdot \left\langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$$

$$= \frac{-16}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \boxed{\frac{-8}{\sqrt{5}}}$$

(b) max. rate of change

$$= |\nabla f(2,1)|$$

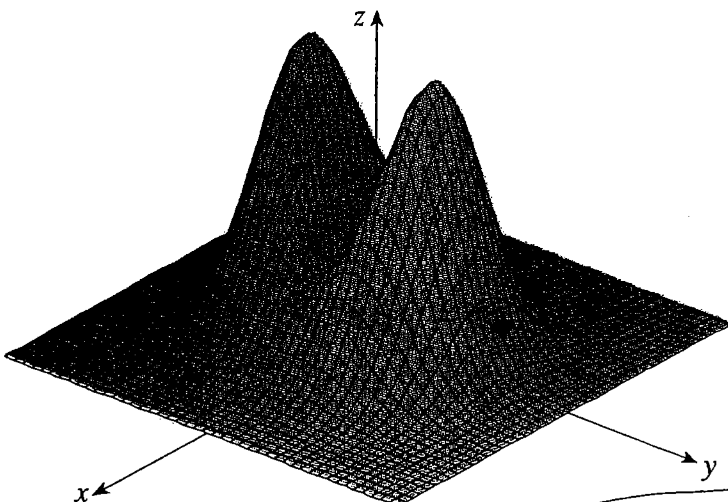
$$= \sqrt{64+64} = \sqrt{128} = \boxed{8\sqrt{2}}$$

direction = direction of $\nabla f(2,1)$

= direction of $\langle 8,-8\rangle$

$$= \boxed{\left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle}$$

3. **(a)** For the function whose graph is shown below, determine the signs of the following partial derivatives: $f_x(0,3)$, $f_y(0,3)$, $f_{xx}(0,3)$, $f_{yy}(0,3)$, $f_{xy}(0,3)$.



$f_x(0,3) = 0$

$f_y(0,3) < 0$

$f_{xx}(0,3) < 0$

$f_{yy}(0,3) > 0$

$f_{xy}(0,3) = 0$

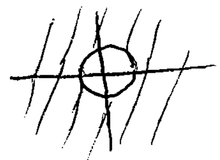
(f_x is 0 along y-axis, so is unchanging as y is increased)

(b) Sketch the domain of the function $f(x,y) = \sqrt{x^2+y^2-1} + \ln(y-x)$. Be sure to indicate which boundary points are in the domain by using dotted/solid lines, and open/closed dots at special points.

domain of $\sqrt{x^2+y^2-1}$ is

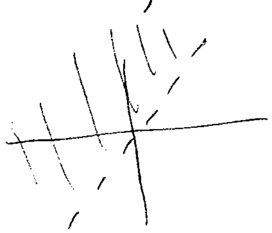
$x^2+y^2-1 \geq 0$, or

$x^2+y^2 \geq 1$

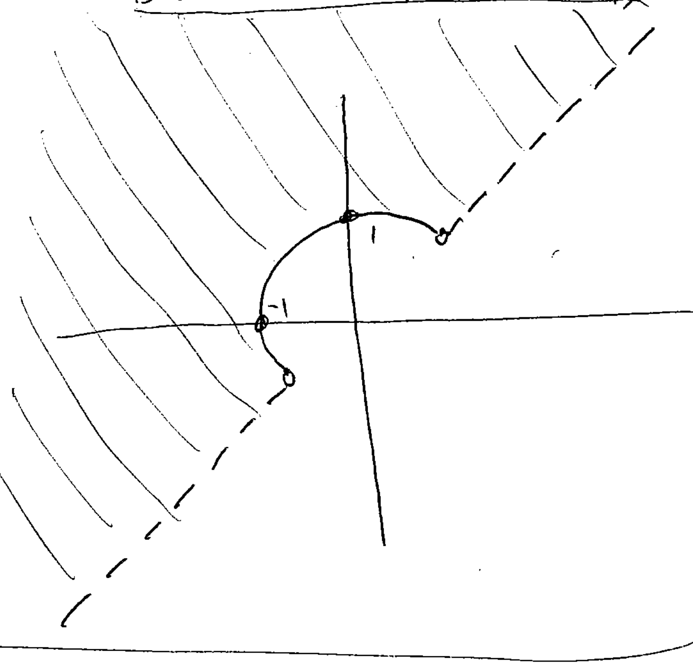


domain of $\ln(y-x)$ is

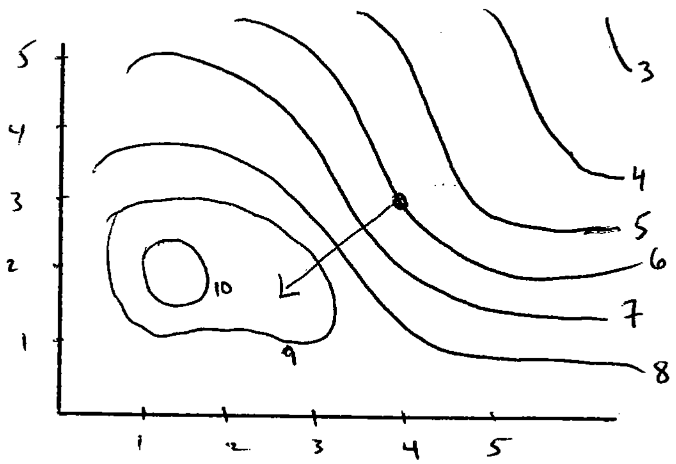
$y-x > 0$, or $y > x$



Domain of $f(x,y)$:



4. **(a)** On the picture below, sketch the gradient vector $\nabla f(4,3)$ for the function f whose level curves are shown. Say how you chose the direction and length of this vector.



direction = $\perp$ to level curve, and "uphill"

length = slope in that direction.

Based on spacing of level curves, this is $\approx \frac{2}{1}$ ("rise" from 6 to 8, "run" ≈ 1)

(b) Recall that a triangle with sides a and b, with angle θ between them, has area $\frac{1}{2}ab\sin(\theta)$. Suppose that the length a is increasing at a rate of 3 inches/sec and the length b is decreasing at a rate of 2 inches/sec. How fast is the area of the triangle changing when $a = 40$ in, $b = 50$ in, and $\theta = \pi/6$?

$A(a,b,\theta) = \frac{1}{2}ab\sin(\theta)$, a, b, θ vary with t

$$\frac{dA}{dt} = \frac{\partial A}{\partial a}\frac{da}{dt} + \frac{\partial A}{\partial b}\frac{db}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt} \quad \text{(chain rule)}$$

$$= \frac{b}{2}\sin(\theta)\frac{da}{dt} + \frac{a}{2}\sin(\theta)\frac{db}{dt} + \frac{ab}{2}\cos(\theta)\frac{d\theta}{dt}$$

At this instant, (θ not changing)

$$= \frac{50}{2}\underbrace{\sin\left(\frac{\pi}{6}\right)}_{1/2}\cdot 3 + \frac{40}{2}\underbrace{\sin\left(\frac{\pi}{6}\right)}_{1/2}\cdot(-2) + \frac{40\cdot 50}{2}\cos\left(\frac{\pi}{6}\right)\cdot 0$$

$$= \frac{150}{4} + \frac{-40}{2} + 0$$

$$= \frac{70}{4}$$

$\boxed{\text{so - area is increasing at a rate of } \frac{70}{4} \text{ inches}^2 \text{ per second.}}$