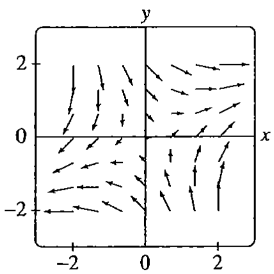
1(a) Match the vector fields ~~to the plots be~~low:

(i) $\mathbf{F}(x,y) = \langle 2, x\rangle$ D

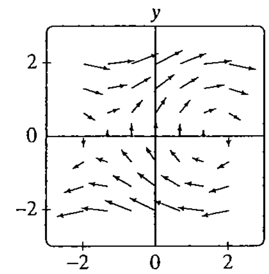
(ii) $\mathbf{F}(x,y) = \langle 2x+2, y\rangle$ C

(iii) $\mathbf{F}(x,y) = \langle y, \cos x\rangle$ B

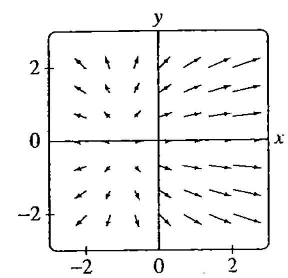
(iv) $\mathbf{F}(x,y) = \langle x+y, x+y\rangle$ A



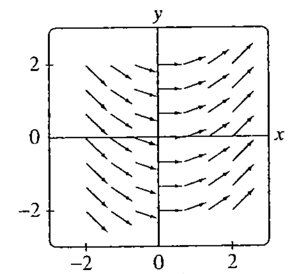
(A)



(B)



(C)



(D)

1(b) Let $\mathbf{F}(x,y,z) = \langle \frac{2xy}{z}, z+\frac{x^2}{z}, y-\frac{x^2y}{z^2}\rangle$. Find a function $f(x,y,z)$ such that $\nabla f = \mathbf{F}$. Then, use the fundamental theorem of line integrals to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = \langle t+1, t^2+2, t^3+3\rangle$, $0 \le t \le 1$.

$$f_x = \frac{2xy}{z} \Rightarrow f = \frac{x^2y}{z} + C(y,z)$$

$$f_y = z + \frac{x^2}{z} \Rightarrow f = yz + \frac{x^2y}{z} + D(x,z)$$

$$f_z = y - \frac{x^2y}{z^2} \Rightarrow f = yz + \frac{x^2y}{z} + E(x,y)$$

So $\boxed{f(x,y,z) = yz + \frac{x^2y}{z} + K}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$= f(2,3,4) - f(1,2,3)$$

$$= 3\cdot 4 + \frac{4\cdot 3}{4} + K - \left(2\cdot 3 + \frac{2}{3} + K\right)$$

$$= \boxed{8\tfrac{1}{3}}$$

2. Evaluate the line integrals:

(a) $\int_C (2x + 9z)ds$ where C is given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$.

$$r'(t) = \langle 1, 2t, 3t^2 \rangle, \quad |r'(t)| = \sqrt{1 + 4t^2 + 9t^4}$$

$$\int_C (2x + 9z)ds = \int_0^1 (2t + 9t^3)\sqrt{1 + 4t^2 + 9t^4}\, dt$$

$$u = 1 + 4t^2 + 9t^4$$
$$du = (8t + 36t^3)\,dt$$

$$\text{integral} = \int_1^{14} \frac{1}{4}\sqrt{u}\, du = \frac{1}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^{14}$$

$$= \boxed{\frac{1}{6}\left(14^{3/2} - 1\right)}$$

(b) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is given by $\mathbf{r}(t) = \langle t^3, -t^2, t \rangle$, $0 \le t \le 1$, and $\mathbf{F}(x, y, z) = \langle \sin x, \cos y, xz \rangle$.

$$r'(t) = \langle 3t^2, -2t, 1 \rangle$$

$$\int_C F \cdot dr = \int_0^1 \langle \sin(t^3), \cos(-t^2), t^4 \rangle \cdot \langle 3t^2, -2t, 1 \rangle\, dt$$

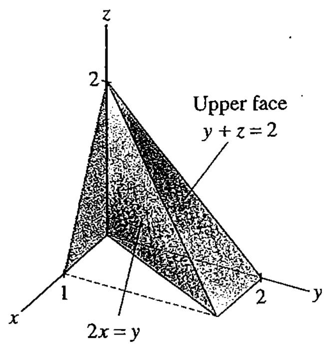
$$= \int_0^1 3t^2 \sin(t^3) - 2t\cos(-t^2) + t^4\, dt$$

$$= -\cos(t^3) + \sin(-t^2) + \frac{1}{5}t^5 \Big|_0^1$$

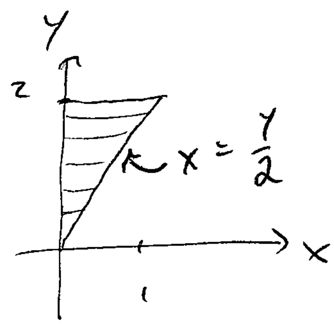
$$= -\cos(1) + \sin(-1) + \frac{1}{5} + \underbrace{\cos(0)}_{=1}$$

$$= \boxed{-\cos(1) + \sin(-1) + \frac{6}{5}}$$

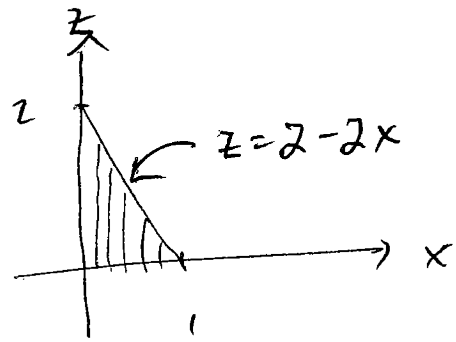
3. The figure below shows the region E bounded by $y+z=2$, $2x=y$, $x=0$, and $z=0$. Express $\iiint_E xe^z\,dV$ in three ways, using $dV=dzdxdy$, $dV=dydzdx$, and $dV=dxdydz$.



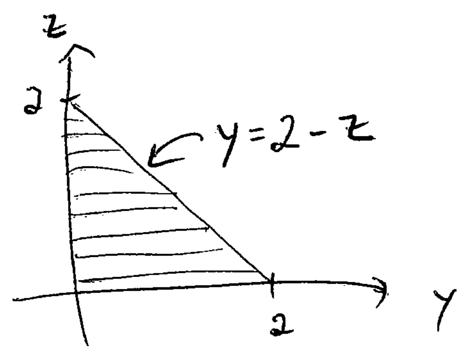
$$\int_0^2\int_0^{\frac{y}{2}}\int_0^{2-y} xe^z\,dz\,dx\,dy$$



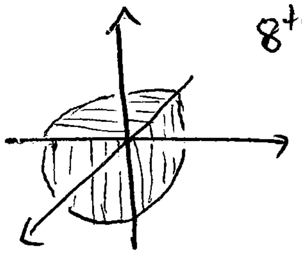
$$\int_0^1\int_0^{2-2x}\int_{2x}^{2-z} xe^z\,dy\,dz\,dx$$



$$\int_0^2\int_0^{2-z}\int_0^{\frac{y}{2}} xe^z\,dx\,dy\,dz$$



4(a) Express the integral $\iiint_E y\,dV$ in spherical coordinates, where E is the region $x^2+y^2+z^2 \le 1$, $x, y, z \le 0$.



8th octant

$y = \rho \sin\varphi \sin\theta$

$dV = \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$

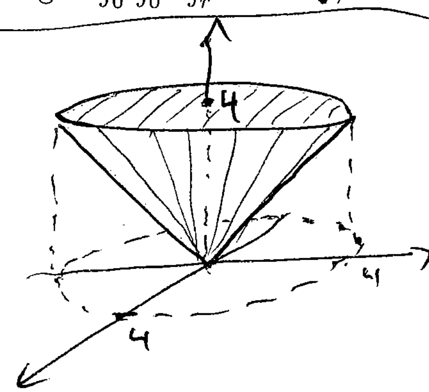
$$\boxed{\int_{\frac{\pi}{2}}^{\pi}\int_{\pi}^{\frac{3\pi}{2}}\int_{0}^{1} (\rho \sin\varphi \sin\theta)\rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi}$$

4(b) Sketch the solid whose volume is given by the integral $\int_0^4 \int_0^{2\pi} \int_r^4 r\,dz\,d\theta\,dr$, and evaluate this integral.

$r \le z \le 4$

$z = r$, cone

$z = 4$, plane



$$\int_0^4 \int_0^{2\pi} rz\Big|_r^4 \, d\theta\, dr = \int_0^4 \int_0^{2\pi} 4r - r^2 \, d\theta\, dr$$

$$= \int_0^{2\pi} d\theta \int_0^4 (4r - r^2)\,dr = 2\pi\left(2r^2 - \frac{r^3}{3}\Big|_0^4\right)$$

$$= 2\pi\left(32 - \frac{64}{3}\right) = \boxed{\frac{64\pi}{3}}$$