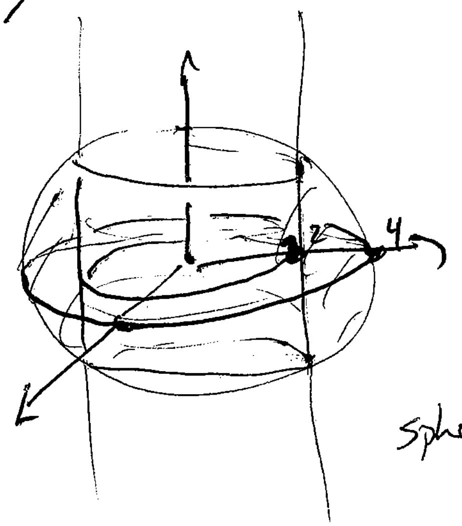
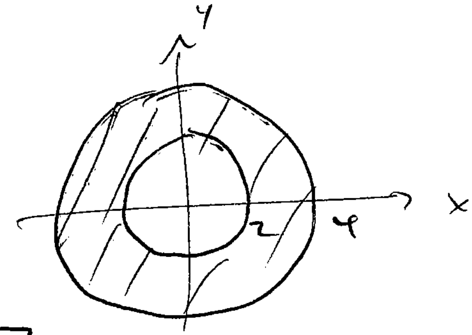
1. Find the volume of the solid region inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$. [Draw a picture.]

radius 2

sphere radius 4



Region:



sphere: $z = \sqrt{16 - x^2 - y^2}$
$= \sqrt{16 - r^2}$

$$\text{Vol} = 2 \cdot \int_0^{2\pi} \int_2^4 \sqrt{16 - r^2}\, r\, dr\, d\theta$$

top half and bottom half

$$= 2 \int_0^{2\pi} \int_{12}^{0} -\tfrac{1}{2}\sqrt{u}\, du\, d\theta \qquad \left[\begin{array}{l} u = 16 - r^2 \\ du = -2r\,dr \end{array}\right]$$

$$= 2 \int_0^{2\pi} -\tfrac{1}{2} \cdot \tfrac{2}{3} u^{3/2} \Big|_{12}^{0} \, d\theta$$

$$= 2 \int_0^{2\pi} \tfrac{1}{3}(12)^{3/2}\, d\theta = \frac{4\pi}{3}(12)^{3/2}$$

$$= \frac{4\pi}{3}(24\sqrt{3})$$

$$= \boxed{32\sqrt{3}\,\pi}$$

2(a) Suppose we want to find the point on the surface $xy^2z^3 = 2$ that is closest to the origin. Write down a system of equations whose solution will include this closest point. [Do not solve the system or proceed any further.]

distance $= \sqrt{x^2+y^2+z^2}$

Minimize $f(x,y,z) = \text{dist}^2 = x^2+y^2+z^2$

subject to $g(x,y,z) = xy^2z^3 = 2$

Lagrange Multipliers: $\nabla f = \lambda \nabla g$:

$$\boxed{\begin{aligned} 2x &= \lambda y^2 z^3 \\ 2y &= \lambda 2xyz^3 \\ 2z &= \lambda 3xy^2z^2 \\ xy^2z^3 &= 2 \end{aligned}}$$

2(b) Find all critical points of the function $f(x, y) = 4xy^2 - x^2y^2 - xy^3$. Then, what can you say about the location of the absolute maximum of $f(x, y)$ on the rectangle $[0, 4] \times [0, 2]$?

$f_x = 4y^2 - 2xy^2 - y^3 = y^2(4-2x-y) = 0$ ①

$f_y = 8xy - 2x^2y - 3xy^2 = xy(8-2x-3y) = 0$ ②

By ①, $y=0$ or $4-2x-y=0$, $y = 4-2x$

If $y=0$ then ② is already satisfied, so any $(x,0)$ is a C.P.

Say $y \neq 0$. Then $y = 4-2x$. ② $\Rightarrow$ $x=0$ or $8-2x-3y=0$

$x=0 \Rightarrow y=4$ so $(0,4)$.

$8-2x-3y=0 \Rightarrow x=1, y=2$ so $(1,2)$.

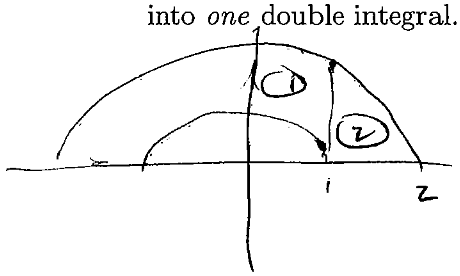
$\boxed{\text{CPs } (x,0), (0,4), (1,2)}$

$\boxed{\text{The abs. max is at a C.P. or is on the boundary.}}$

3(a) Use polar coordinates to combine the sum

$$\underbrace{\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} xy\, dy\, dx}_{1} + \underbrace{\int_1^2 \int_0^{\sqrt{4-x^2}} xy\, dy\, dx}_{2}$$

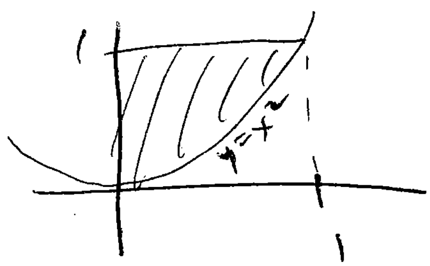
into *one* double integral. Do not evaluate the integral. [Draw the region!]



In polar coords:

$$\boxed{\int_0^{\pi/2} \int_1^2 r\cos\theta \, r\sin\theta \, r\, dr\, d\theta}$$

3(b) Sketch the region, and evaluate the integral $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$ by reversing the order of integration.



$$\int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy$$

$$= \int_0^1 \frac{1}{4} x^4 \sin(y^3) \Big|_0^{\sqrt{y}} dy = \int_0^1 \frac{1}{4} y^2 \sin(y^3)\, dy$$

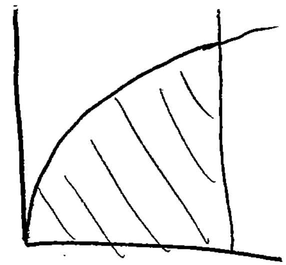
$$= \int_0^1 \frac{1}{12} \sin(u)\, du \qquad \begin{bmatrix} u = y^3 \\ du = 3y^2\, dy \end{bmatrix}$$

$$= \frac{-1}{12} \cos(u) \Big|_0^1 = \boxed{\frac{-1}{12}(\cos(1) - 1)}$$

4. A lamina occupying the region D bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$ has density given by $\rho(x, y) = x$.

(a) Find the mass m of the lamina.

(b) Find $\bar{x}$ and $\bar{y}$ (the center of mass of the lamina).



(a) $$m = \int_0^1 \int_0^{\sqrt{x}} x \, dy \, dx$$

$$= \int_0^1 x\sqrt{x} \, dx = \frac{2}{5} x^{5/2} \Big|_0^1 = \boxed{\frac{2}{5}}$$

(b) $$\bar{x} = \frac{5}{2} \int_0^1 \int_0^{\sqrt{x}} x^2 \, dy \, dx$$

$$= \frac{5}{2} \int_0^1 x^2 \sqrt{x} \, dx = \frac{5}{2} \cdot \frac{2}{7} x^{7/2} \Big|_0^1 = \boxed{\frac{5}{7}}$$

$$\bar{y} = \frac{5}{2} \int_0^1 \int_0^{\sqrt{x}} xy \, dy \, dx = \frac{5}{2} \int_0^1 \frac{1}{2} xy^2 \Big|_0^{\sqrt{x}} dx$$

$$= \frac{5}{2} \int_0^1 \frac{1}{2} x^2 \, dx = \frac{5}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} x^3 \Big|_0^1$$

$$= \boxed{\frac{5}{12}}$$