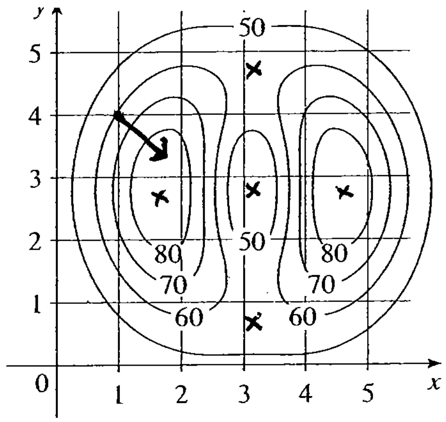
**1(a)** A contour map is given for the function $f(x,y)$. Estimate $f_x(1,4)$ and $f_y(1,4)$. Also, draw a unit vector at $(1,4)$ indicating the direction of the gradient vector at that point.

Are there any critical points in the region shown? If so, where?




$$\boxed{\begin{aligned} f_x(1,4) &\approx 20 \\ f_y(1,4) &\approx -18 \end{aligned}}$$

Critical points marked by " × " (5 of them)

**(b)** Find the local maxima, local minima, and saddle points for the function $g(x,y) = (x^2+y^2)e^{-x}$.

$g_x = (x^2+y^2)(-e^{-x}) + 2xe^{-x} = (2x - x^2 - y^2)e^{-x}$

$g_y = 2ye^{-x}$. $\qquad g_y = 0 \Rightarrow y = 0$.

$\qquad g_x = 0 \Rightarrow 2x - x^2 = 0 \Rightarrow x = 0, 2$.

<u>critical points</u>; $(0,0)$ and $(2,0)$.

$g_{xx} = (2x - x^2 - y^2)(-e^{-x}) + (2-2x)e^{-x}$

$\quad = (x^2+y^2-4x+2)e^{-x}$

$g_{yy} = 2e^{-x}$ $\qquad D = g_{xx}g_{yy} - (g_{xy})^2$

$g_{yx} = -2ye^{-x}$

$D(0,0) = 2 \cdot 2 - 0 = \underline{4}$, $\underline{g_{xx}(0,0) = 2} \Rightarrow \boxed{\text{local min at } (0,0)}$

$D(2,0) = \underline{-4e^{-4}} < 0 \Rightarrow \boxed{\text{Saddle point at } (2,0)}$

**2(a)** Is $\mathbf{G}(x,y,z) = \langle x^2yz, -xy^2z, xyz^2\rangle$ the curl of a vector field $\mathbf{F}$? If so, find $\mathbf{F}$, and if not, say why not.

$\operatorname{div} \underline{G} = 2xyz - 2xyz + 2xyz = \underline{2xyz}.$

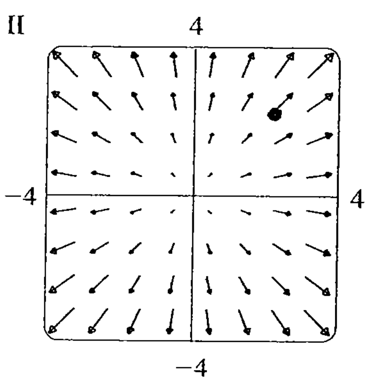
- not $\underline{\text{zero}}$.

If $\underline{G} = \operatorname{curl} \underline{F}$ then $\operatorname{div}(\operatorname{curl} \underline{F}) = 0$

but $\operatorname{div}(\underline{G}) \neq 0$

so $\underline{G}$ cannot be $\operatorname{curl} \underline{F}$ for any $\underline{F}$.

**(b)** The picture below shows the vector field $\mathbf{F}(x,y) = \langle P, Q\rangle$. Say whether the following quantities are positive, negative, or zero, at the point $(2,2)$: $P_x$, $P_y$, $Q_x$, $Q_y$, $\operatorname{curl}\mathbf{F}\cdot\mathbf{k}$, $\operatorname{div}\mathbf{F}$.



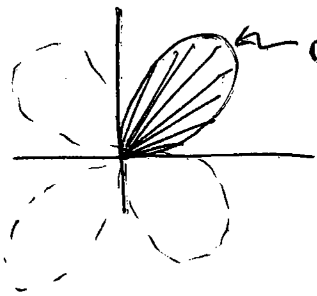

$P_x > 0$ $\qquad P_y = 0$

$Q_x = 0$ $\qquad Q_y > 0$

$\operatorname{curl} \underline{F} \cdot \underline{k} = Q_x - P_y = 0$
(no rotation)

$\operatorname{div} \underline{F} = P_x + Q_y > 0$
(more leaving than entering)

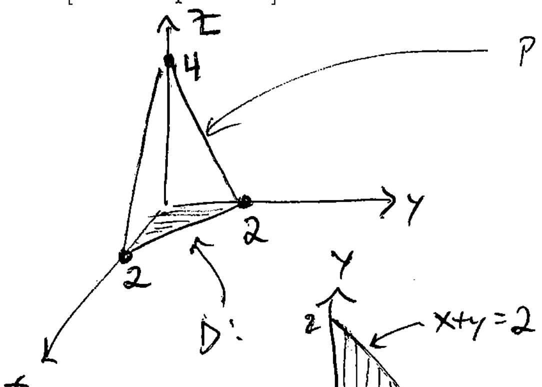
**3(a)** Write down an integral which represents the area of one lobe of the polar curve $r = \sin(2\theta)$. [Draw a picture.]




Area is $\iint_D 1\,dA$

$$= \boxed{\int_0^{\pi/2}\int_0^{\sin(2\theta)} r\,dr\,d\theta}$$

**(b)** Evaluate $\iiint_E y\,dV$ where $E$ is bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x+2y+z = 4$. [Draw a picture.]




plane $z = 4-2x-2y$

$$\int_0^2\int_0^{2-x}\int_0^{4-2x-2y} y\,dz\,dy\,dx$$

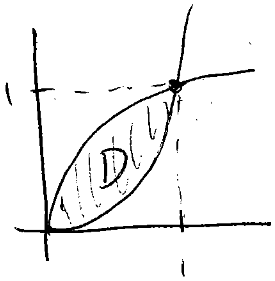
$$= \int_0^2\int_0^{2-x} 4y - 2xy - 2y^2\,dy\,dx$$

$$= \int_0^2 2y^2 - xy^2 - \tfrac{2}{3}y^3\Big|_0^{2-x} dx = \int_0^2 (2-x)(2-x)^2 - \tfrac{2}{3}(2-x)^3\,dx = \int_0^2 \tfrac{1}{3}(2-x)^3\,dx$$

$$= \int_2^0 -\tfrac{1}{3}u^3\,du \quad \left(\begin{array}{l} u = 2-x \\ du = -dx \end{array}\right)$$

$$= \int_0^2 \tfrac{1}{3}u^3\,du = \tfrac{1}{12}u^4\Big|_0^2 = \tfrac{16}{12} = \boxed{\tfrac{4}{3}}$$

4(a) Use Green's Theorem to evaluate $\int_C (y + e^{\sqrt{x}})dx + (2x + \cos(y^2))dy$ where $C$ is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.



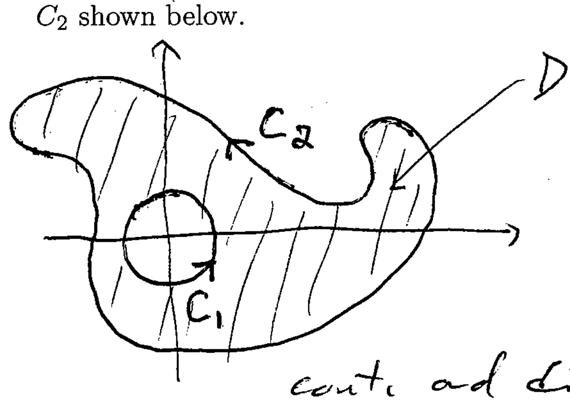

Greens Thm: Integral above is equal to $\iint_D \left( \frac{\partial}{\partial x}(2x+\cos(y^2)) - \frac{\partial}{\partial y}(y+e^{\sqrt{x}}) \right) dA$

$= \iint_D (2-1)\, dA = \int_0^1 \int_{y^2}^{\sqrt{y}} dx\, dy$

$= \int_0^1 \sqrt{y} - y^2\, dy = \frac{2}{3} y^{3/2} - \frac{1}{3} y^3 \Big|_0^1$

$= \boxed{\frac{1}{3}}$

(b) Consider the vector field $\mathbf{F}(x, y) = \langle P, Q \rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$, which is undefined at the origin. The partial derivatives of $P$ and $Q$ are given by $P_x = \frac{2xy}{(x^2+y^2)^2}$, $P_y = \frac{y^2-x^2}{(x^2+y^2)^2}$, $Q_x = \frac{y^2-x^2}{(x^2+y^2)^2}$, $Q_y = \frac{-2xy}{(x^2+y^2)^2}$. Using Green's Theorem, explain why $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for the curves $C_1$ and $C_2$ shown below.

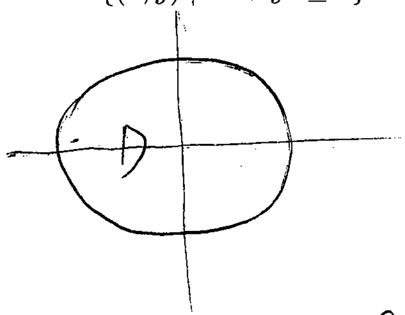



Let D be the region between $C_1$ and $C_2$. Then the oriented boundary of D is $C_2 - C_1$. Since F is cont. and diff. on D, Green's Theorem says $\iint_D (Q_x - P_y)\, dA = \int_{C_2} F \cdot dr + \int_{-C_1} F \cdot dr$

$= \int_{C_2} F \cdot dr - \int_{C_1} F \cdot dr$.

Since $(Q_x - P_y) = 0$, this difference is also zero.

**5.** Find the absolute maximum and minimum of the function $f(x, y) = 8x^2 - 3y^2$ on the set $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.




Critical Points Inside:

$f_x = 16x = 0 \Rightarrow x = 0$

$f_y = -6y = 0 \Rightarrow y = 0$

So $(0,0)$ only.

Boundary: we have $y^2 = 1 - x^2$ so on bdry,

$f(x,y) = 8x^2 - 3(1-x^2) = 11x^2 - 3, \quad -1 \leq x \leq 1.$

Say $g(x) = 11x^2 - 3$.

$g'(x) = 22x = 0 \Rightarrow x = 0$ is the only crit. point.

So $(0, \pm 1)$ should be considered.

Also, endpoints $x = \pm 1$ give $(\pm 1, 0)$.

Now Compare

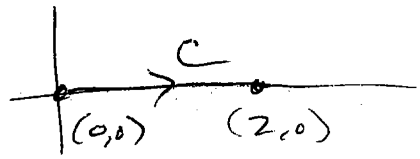
$f(0,0) = 0$

$f(\pm 1, 0) = 8$

$f(0, \pm 1) = -3$

$$\boxed{\begin{array}{l} \text{Abs. max} = 8 \\ \text{Abs. min} = -3 \end{array}}$$

**6. (a)** A certain function $f(x, y)$ has gradient $\langle xe^{x^2}, \sin(y^2)\rangle$. Its value at $(0,0)$ is $1/2$. Using the fundamental theorem of line integrals, find $f(2,0)$. [Note: you will not be able to find a formula for $f$.]




$C$ given by $\underline{r}(t) = \langle t, 0\rangle$, $0 \le t \le 2$
then $\underline{r}'(t) = \langle 1, 0\rangle$.

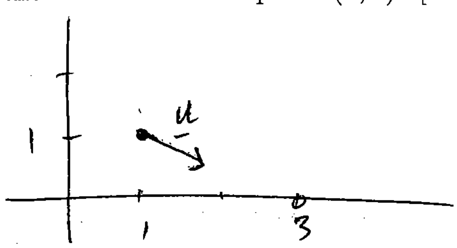
FTLI: $f(\underline{r}(b)) - f(\underline{r}(a)) = \int_C \nabla f \cdot d\underline{r}$

So $f((2,0)) - \frac{1}{2} = \int_0^2 \langle te^{t^2}, \sin(0)\rangle \cdot \langle 1, 0\rangle\, dt$

$= \int_0^2 te^{t^2}\, dt = \frac{1}{2}e^{t^2}\Big|_0^2 = \frac{1}{2}e^4 - \frac{1}{2}$.

$\Rightarrow f(2,0) = \boxed{\frac{1}{2}e^4}$

**(b)** Use the gradient to find the directional derivative of $g(x, y) = x^2y - 4xy^3$ at the point $(1,1)$ in the direction of the point $(3,0)$. [Draw a picture.]




$\underline{u} = \frac{\langle 2, -1\rangle}{|\langle 2, -1\rangle|} = \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$.

$\nabla g = \langle 2xy - 4y^3, x^2 - 12xy^2\rangle$

$\nabla g(1,1) = \langle -2, -11\rangle$

$D_{\underline{u}}g = \nabla g \cdot \underline{u} = \langle -2, -11\rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$

$= \frac{-4}{\sqrt{5}} + \frac{11}{\sqrt{5}} = \boxed{\frac{7}{\sqrt{5}}}$