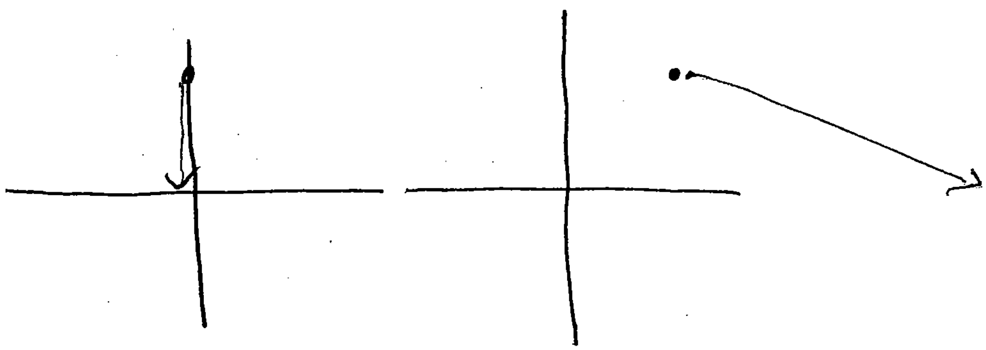
1(a) Let $\mathbf{F}(x,y) = x^2\mathbf{i} - y\mathbf{j}$. True or False:

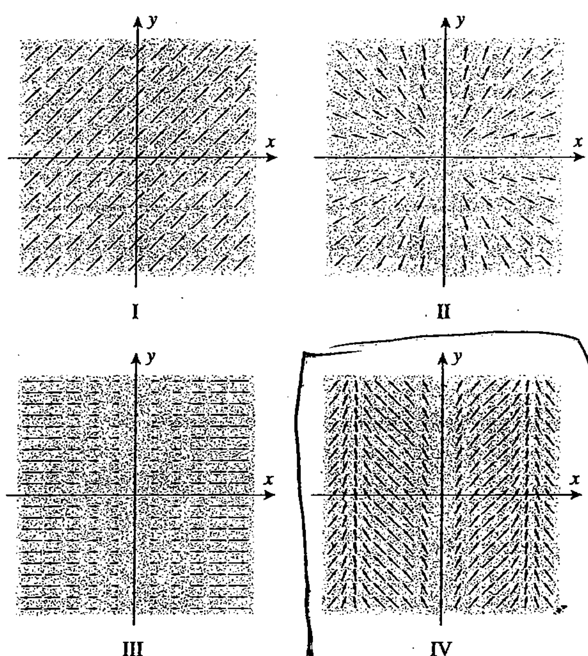
(i) $|\mathbf{F}(x,y)| \to 0$ as $(x,y) \to (0,0)$. TRUE

(ii) If (x,y) is on the positive y-axis, then the vector points in the negative y-direction. TRUE

(iii) If (x,y) is in the first quadrant, then the vector points down and to the right. TRUE



(b) One of the vector fields below is $\mathbf{F}(x,y) = \langle \sin x, 1\rangle$. Find possible formulas for the other three, and indicate which is which.



I: $\langle 1, 1\rangle$

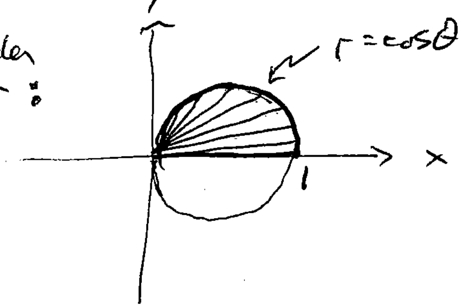
II: $\dfrac{\langle x, y\rangle}{\sqrt{x^2+y^2}}$

III: $\langle x, 0\rangle$

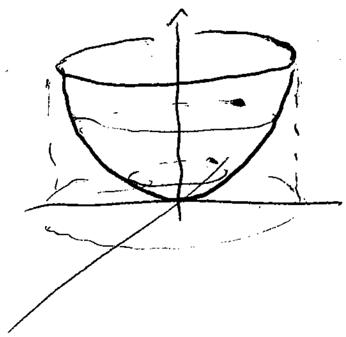
IV: $\langle \sin x, 1\rangle$

2(a) Suppose $\iiint_E f(z,r,\theta)dV$ is the iterated integral $\int_0^{\pi/2}\int_0^{\cos\theta}\int_0^{r^2} r\sin\theta\, dz\, dr\, d\theta$. What is $f(z,r,\theta)$? Sketch and describe carefully the region E.

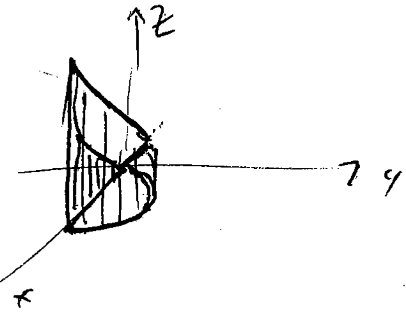
bounded by: $\left\{ \begin{array}{ll} z=0, & z=r^2 \\ r=0, & r=\cos\theta \\ \theta=0, & \theta=\pi/2. \end{array} \right\} \to$ cylinder over:



paraboloid

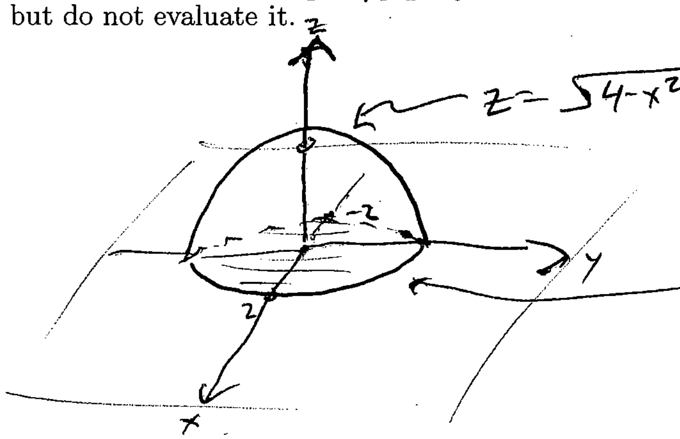


So E looks like:
(it's the region below the paraboloid and above the half-circle)



2(b) Convert the integral $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$ to spherical coordinates, but do not evaluate it.

$z=\sqrt{4-x^2-y^2}$, hemisphere of radius 2



$y=\pm\sqrt{4-x^2}$, circle of radius 2

Spherical Coords: $dz\,dy\,dx \to \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$

$\sqrt{x^2+y^2+z^2} \to \rho$

$z^2 \to \rho^2\cos^2\varphi$

$$\boxed{\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2} \rho^5\cos^2\varphi\sin\varphi\, d\rho\, d\theta\, d\varphi}$$

3(a) A certain function $f(x,y)$ has gradient $\langle xe^{(x^2)}, \cos(y^2)\rangle$. Its value at $(0,0)$ is $1/2$. Find its value at $(1,0)$. [Note: you won't be able to find $f(x,y)$ directly.]

Use: $f(1,0) - f(0,0) = \int_C \nabla f \cdot d\underline{r}$, C = any curve from $(0,0)$ to $(1,0)$

try C given by: $\underline{r}(t) = \langle t, 0\rangle \Rightarrow \underline{r}'(t) = \langle 1, 0\rangle$ $(0 \le t \le 1)$

$$\int_C \nabla f \cdot d\underline{r} = \int_0^1 \langle xe^{x^2}, \cos(y^2)\rangle \cdot \langle 1, 0\rangle \, dt$$

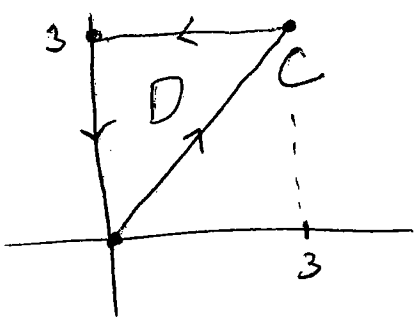
$$= \int_0^1 \langle te^{t^2}, \cos(0)\rangle \cdot \langle 1, 0\rangle \, dt$$

$$= \int_0^1 te^{t^2} dt \qquad u = t^2, \; du = 2t\,dt$$

$$= \frac{1}{2}\int_0^1 e^u du = \frac{1}{2}(e-1) = \frac{e}{2} - \frac{1}{2}.$$

So $f(1,0) = \left(\frac{e}{2} - \frac{1}{2}\right) + f(0,0) = \boxed{\frac{e}{2}}$

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle \underbrace{y + \cos x \sin y}_{P},\ \underbrace{y + \sin x \cos y}_{Q} \rangle$ and C is the triangle from $(0,0)$ to $(3,3)$ to $(0,3)$ to $(0,0)$.



Green's Theorem: $\int_C \underline{F} \cdot d\underline{r} = \iint_D (Q_x - P_y)\, dA$

$Q_x = \cos x \cos y \qquad P_y = 1 + \cos x \cos y$

$Q_x - P_y = -1$

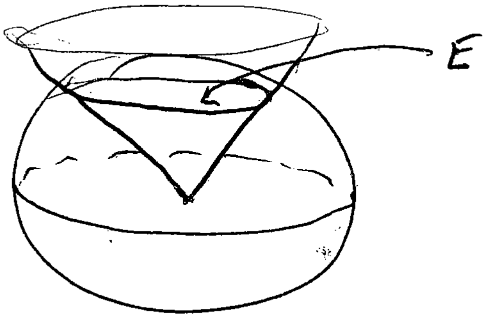
$$\iint_D -1 \, dA = \int_0^3 \int_x^3 -1 \, dy\, dx = \int_0^3 \left[-y\right]_x^3 dx$$

$$= \int_0^3 (-3+x)\, dx = -3x + \tfrac{1}{2}x^2 \Big|_0^3$$

$$= \boxed{-\frac{9}{2}}$$

(Alternatively: $\iint_D -1\, dA = -\text{Area}(D) = -\frac{9}{2}$.)

4. Find the mass of the solid that is enclosed by the sphere $x^2 + y^2 + z^2 = 1$ and lies within the cone $z = \sqrt{x^2 + y^2}$ if the density is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.



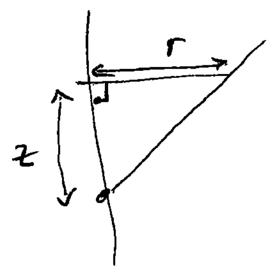
E is given by: $0 \le \rho \le 1$

$0 \le \theta \le 2\pi$

$0 \le \varphi \le \underline{?}$

Cone $z^2 = x^2 + y^2$ is also $z^2 = r^2$, or $z = \pm r$

so:



is a 90-45-45 triangle

i.e. angle from vertical

is $\pi/4$.

So $\underline{0 \le \varphi \le \pi/4}$.

$$\text{Mass} = \iiint_E \sqrt{x^2+y^2+z^2}\, dV$$

$$= \int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho \cdot \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \int_0^1 \rho^3 d\rho \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\pi/4} \sin\varphi\, d\varphi$$

$$= \left(\tfrac{1}{4}\rho^4 \Big|_0^1 \right) \cdot \left(\theta \Big|_0^{2\pi} \right) \cdot \left(-\cos\varphi \Big|_0^{\pi/4} \right)$$

$$= \frac{2\pi}{4}\left(-\frac{\sqrt{2}}{2} + 1 \right)$$

$$= \boxed{\frac{\pi}{2}\left(1 - \frac{\sqrt{2}}{2}\right)}$$

5(a) Are the following vector fields **F** conservative? If so, find a function f with $\nabla f = \mathbf{F}$, and otherwise give a reason why not.

(i) $\mathbf{F}(x, y) = \langle e^x \cos y, -e^x \sin y \rangle$

(ii) $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$

(i) $Q_x = -e^x \sin y$, $P_y = -e^x \sin y \Rightarrow \boxed{\text{Conservative}}$

$f_x = e^x \cos y \Rightarrow f = e^x \cos y + C(y)$

$f_y = -e^x \sin y \Rightarrow f = e^x \cos y + D(x)$

So $\boxed{f(x,y) = e^x \cos y}$ works.

(ii) $Q_x = x(ye^{xy}) + e^{xy}$, $P_y = y(xe^{xy}) + e^{xy} \Rightarrow \boxed{\text{conservative}}$

$f_x = ye^{xy} \Rightarrow f = y \cdot \frac{1}{y} e^{xy} + C(y) = e^{xy} + C(y)$

$f_y = xe^{xy} \Rightarrow f = e^{xy} + D(x)$

So $\boxed{f(x,y) = e^{xy}}$ works.

(b) Explain why the integral $\int_{(0,0)}^{(3,2)} 2xe^y\,dx + x^2e^y\,dy$ is independent of path, and find its value.

This is the line integral of $\underline{F} = \langle \underset{P}{2xe^y}, \underset{Q}{x^2e^y} \rangle$, which is conservative since $Q_x = 2xe^y = P_y$. $\Rightarrow$ Indep. of path.

Now $\underline{F} = \nabla f$, so $f_x = 2xe^y \Rightarrow f = x^2e^y + C(y)$

$f_y = x^2e^y \Rightarrow f = x^2e^y + D(x)$

$f(x,y) = x^2e^y + C$ works.

Now $\int_{(0,0)}^{(3,2)} \nabla f \cdot d\mathbf{r} = f(3,2) - f(0,0)$

$= 9e^2 - 0$

$= \boxed{9e^2}$

6. Evaluate the following line integrals.

(a) $\int_C y\,dx + z\,dy - x\,dz$ where C is the line segment from $(0,0,0)$ to $(1,1,1)$.

(b) $\int_C 3x^2yz\,ds$ where C is given by $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3\rangle$, $0 \le t \le 1$.

(a) Use $r(t) = \langle t,t,t\rangle$, $0 \le t \le 1$, so $r'(t) = \langle 1,1,1\rangle$

$$\int_C y\,dx + z\,dy - x\,dz = \int_0^1 t\,dt + \int_0^1 t\,dt - \int_0^1 t\,dt$$

$(dx = dt,\ dy = dt,\ dz = dt)$

$$= \int_0^1 t\,dt = \frac{1}{2}t^2\Big|_0^1 = \boxed{\frac{1}{2}}.$$

(b) $x'(t) = 1$, $y'(t) = 2t$, $z'(t) = 2t^2$

$$ds = \sqrt{1^2 + (2t)^2 + (2t^2)^2}\,dt = \sqrt{1 + 4t^2 + 4t^4}\,dt$$
$$= \sqrt{(1+2t^2)^2}\,dt$$
$$= (1+2t^2)\,dt$$

$$\int_C 3x^2yz\,ds = \int_0^1 3t^2 \cdot t^2 \cdot \frac{2}{3}t^3 (1+2t^2)\,dt$$

$$= \int_0^1 2t^7 + 4t^9\,dt = \frac{1}{4}t^8 + \frac{2}{5}t^{10}\Big|_0^1$$

$$= \frac{1}{4} + \frac{2}{5}$$

$$= \boxed{\frac{13}{20}}$$