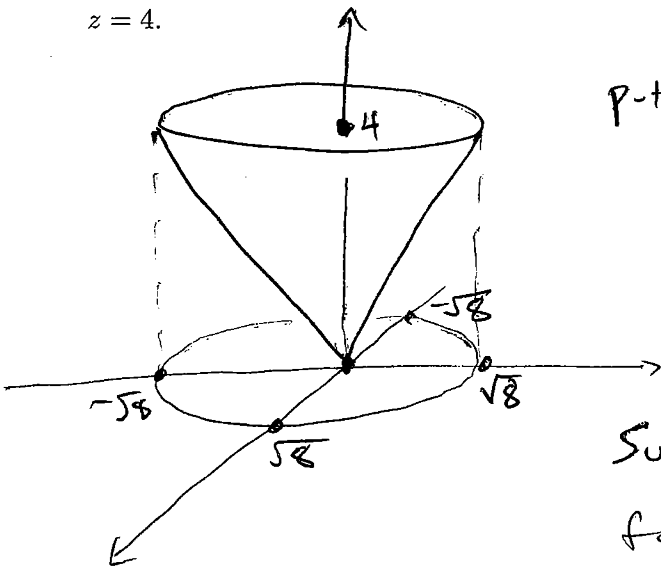
1. Find the area of the part of the cone $z^2 = 2(x^2+y^2)$ which lies between the planes $z=0$ and $z=4$.



put $z=4$ in: $16 = 2(x^2+y^2)$

$8 = x^2+y^2$

D = circle of radius $\sqrt{8}$.

Surface Area $= \iint_D \sqrt{1+f_x^2+f_y^2}\, dA$

for the graph of $f(x,y)$

$$z = f(x,y) = \sqrt{2(x^2+y^2)}$$

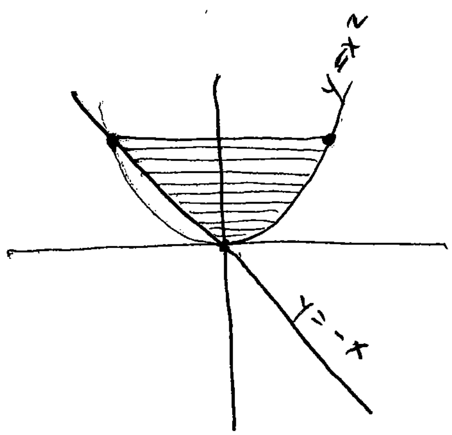
$$f_x = \frac{2x}{\sqrt{2(x^2+y^2)}}, \quad f_y = \frac{2y}{\sqrt{2(x^2+y^2)}}$$

$$1+f_x^2+f_y^2 = \frac{2(x^2+y^2)+4x^2+4y^2}{2(x^2+y^2)} = \frac{6(x^2+y^2)}{2(x^2+y^2)} = 3$$

$$SA = \iint_D \sqrt{3}\, dA = \sqrt{3}\iint_D dA = \sqrt{3}(\text{area of } D)$$

$$= \boxed{\sqrt{3}\cdot 8\pi}$$

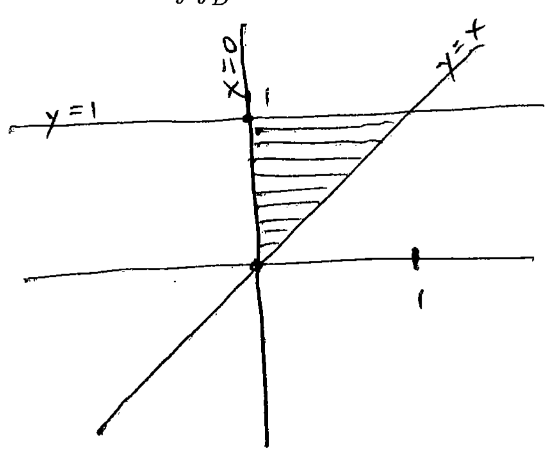
2a. Sketch the region, and change the order of integration for $\int_0^1 \int_{-y}^{\sqrt{y}} f(x,y)\,dxdy$.



two parts!

$$\boxed{\int_{-1}^{0}\int_{-x}^{1} f(x,y)\,dydx + \int_{0}^{1}\int_{x^2}^{1} f(x,y)\,dydx}$$

2b. Evaluate $\iint_D \cos(y^2)\,dA$ where D is the region bounded by the lines $y = x$, $x = 0$, and $y = 1$.



$$\int_0^1\int_0^y \cos(y^2)\,dx\,dy$$

$$= \int_0^1 x\cos(y^2)\Big|_0^y \,dy$$

$$= \int_0^1 y\cos(y^2)\,dy = \frac{1}{2}\int_0^1 \cos(u)\,du \quad \left(\begin{array}{l} u = y^2 \\ du = 2y\,dy \end{array}\right)$$

$$= \frac{1}{2}\sin(u)\Big|_0^1 = \frac{1}{2}\sin(1) - \frac{1}{2}\sin(0)$$

$$= \boxed{\frac{1}{2}\sin(1)}$$

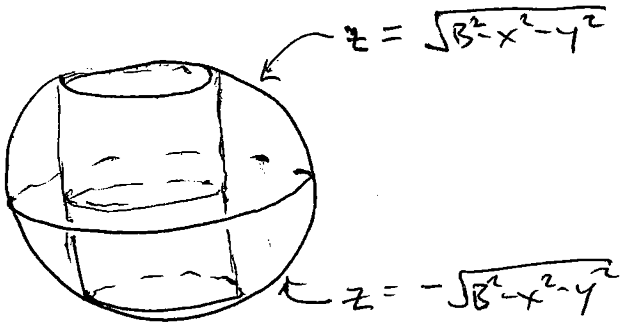
note integrating in the other order won't work.

3. A napkin ring is made from a wooden sphere of radius B by drilling out a cylindrical hole of radius A vertically through its center.

(a) Find the volume of wood left in the ring.

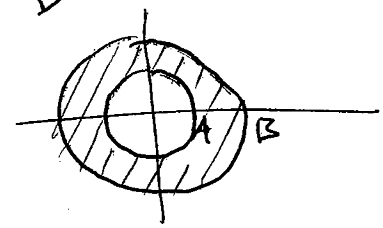
(b) Find the height of the ring in terms of A and B, and express the volume in terms of the height.

a)



Vol. of ring $= 2\iint_D \sqrt{B^2-x^2-y^2}\,dA$

where $D =$



In polar coords,

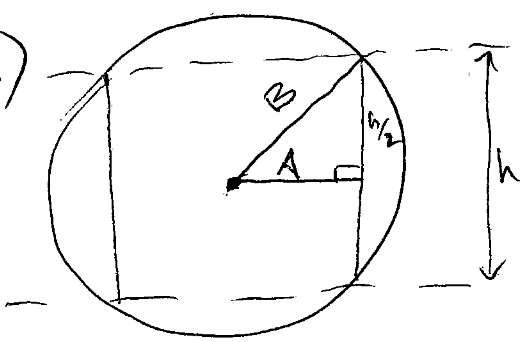
$$\text{Vol} = 2\int_0^{2\pi}\int_A^B \sqrt{B^2-r^2}\; r\,dr\,d\theta \qquad u = B^2-r^2,\quad du = -2r\,dr$$

$$= -\int_0^{2\pi}\int_{B^2-A^2}^{0} \sqrt{u}\,du\,d\theta$$

$$= \int_0^{2\pi} \tfrac{2}{3}(B^2-A^2)^{3/2}\,d\theta = \tfrac{2}{3}(B^2-A^2)^{3/2}\int_0^{2\pi} d\theta$$

$$= \boxed{\frac{4\pi}{3}(B^2-A^2)^{3/2}}$$

b)



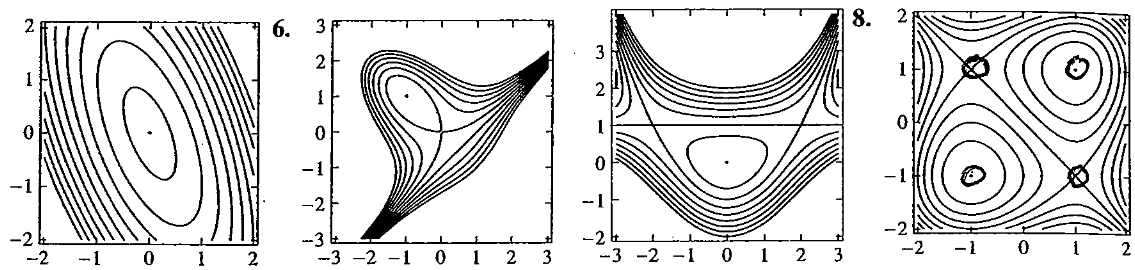
$$\Rightarrow \frac{h}{2} = \sqrt{B^2-A^2}, \text{ or } \boxed{h = 2\sqrt{B^2-A^2}}.$$

Then $\boxed{\text{Vol} = \frac{4\pi}{3}\left(\frac{h}{2}\right)^3}$.

4. Consider the function $f(x,y) = x^3 + y^3 - 3x - 3y$.

(a) Find all critical points, and use the second derivative test to determine the behavior at each of these points.

(b) Identify the contour plot of f from the four possibilities below, and locate the critical points.



a) $f_x = 3x^2 - 3 = 0$ $\qquad$ $f_y = 3y^2 - 3 = 0$

$3x^2 = 3$ $\qquad$ $3y^2 = 3$

$x = \pm 1$ $\qquad$ $y = \pm 1$

$\boxed{\text{critical points: } (1,1), (1,-1), (-1,1), (-1,-1).}$

$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = 0$

$D = f_{xx} f_{yy} - f_{xy}^2 = 36xy.$

$(1,1)$: $D = 36 > 0$, $f_{xx} = 6 > 0$ $\Rightarrow$ local min. at $(1,1)$

$(1,-1)$: $D = -36 < 0$ $\Rightarrow$ saddle point at $(1,-1)$

$(-1,1)$: $D = -36 < 0$ $\Rightarrow$ saddle point at $(-1,1)$

$(-1,-1)$: $D = 36 > 0$, $f_{xx} = -6 < 0$ $\Rightarrow$ local max. at $(-1,-1)$

b) The last picture. Critical points are circled.

5. Consider the following problem: find the points on the surface $z^2 = x^2y^2 + 3x + 1$ that are closest to the point $(1, 1, 1)$.

(a) Write down the function $f(x, y, z)$ to be minimized, and express the constraint as $g(x, y, z) = k$.

(b) Using Lagrange multipliers, write down a system of equations whose solutions include the desired points on the surface. Do not solve the system or proceed any further.

a) Distance to $(1,1,1)$ is given by $d = \sqrt{(x-1)^2+(y-1)^2+(z-1)^2}$.

Minimizing distance is equivalent to minimizing $(\text{distance})^2$, so we'll use:

$$\boxed{\begin{aligned} f(x,y,z) &= (x-1)^2 + (y-1)^2 + (z-1)^2 \\ \text{Constraint: } g(x,y,z) &= z^2 - x^2y^2 - 3x = 1 \end{aligned}}$$

b) $f_x = 2(x-1)$, $f_y = 2(y-1)$, $f_z = 2(z-1)$

$g_x = -2xy^2 - 3$, $g_y = -2x^2y$, $g_z = 2z$

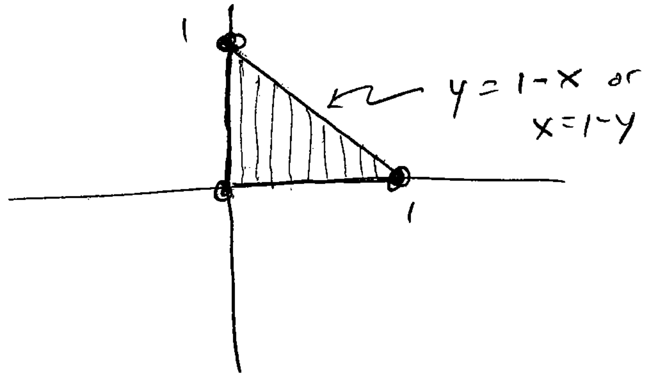
System:

$$\boxed{\begin{aligned} 2(x-1) &= \lambda(-2xy^2-3) \\ 2(y-1) &= \lambda(-2x^2y) \\ 2(z-1) &= \lambda(2z) \\ z^2 - x^2y^2 - 3x &= 1 \end{aligned}}$$

6. A triangular lamina with vertices $(0,0)$, $(0,1)$, and $(1,0)$ has density function $\rho(x,y) = xy$.

(a) Find its mass.

(b) Find its center of mass.



$$m = \int_0^1 \int_0^{1-y} xy\,dx\,dy$$

$$= \int_0^1 \tfrac{1}{2}x^2 y \Big|_0^{1-y} dy$$

$$= \int_0^1 \tfrac{1}{2}(1-y)^2 y\,dy = \int_0^1 \tfrac{1}{2}y - y^2 + \tfrac{1}{2}y^3\,dy$$

$$= \tfrac{1}{4}y^2 - \tfrac{1}{3}y^3 + \tfrac{1}{8}y^4 \Big|_0^1 = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \boxed{\frac{1}{24}}$$

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\int_0^1 \int_0^{1-y} x^2 y\,dx\,dy = 24\int_0^1 \int_0^{1-y} x^2 y\,dx\,dy$$

$$= 24\int_0^1 \tfrac{1}{3}(1-y)^3 y\,dy = 8\int_0^1 y - 3y^2 + 3y^3 - y^4\,dy$$

$$= 8\left(\tfrac{1}{2}y - y^3 + \tfrac{3}{4}y^4 - \tfrac{1}{5}y^5\right)\Big|_0^1$$

$$= 4 - 8 + 6 - \frac{8}{5}$$

$$= \boxed{\frac{2}{5}}$$

$$\bar{y} = \frac{M_x}{m} = \cdots = \boxed{\frac{2}{5}}$$ the calculation is identical, with x and y reversed.

Center of mass $= \boxed{\left(\frac{2}{5}, \frac{2}{5}\right)}$