1. Let $\mathbf{r}(t) = \langle \sin 3t, t, \cos 3t \rangle$.

(a) Find the tangent vector $\mathbf{T}(t)$.

(b) Find the curvature κ at the point $(0, \pi, -1)$.

(c) Find the normal vector $\mathbf{N}(t)$.

a) $r'(t) = \langle 3\cos 3t, 1, -3\sin 3t \rangle$

$|r'(t)| = \sqrt{9\cos^2 3t + 1 + 9\cos^2 3t} = \sqrt{10}$

$$\boxed{T(t) = \frac{r'(t)}{|r'(t)|} = \left\langle \frac{3}{\sqrt{10}}\cos 3t, \frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}}\sin 3t \right\rangle}$$

b) $T'(t) = \left\langle \frac{-9}{\sqrt{10}}\sin 3t, 0, \frac{-9}{\sqrt{10}}\cos 3t \right\rangle$

$|T'(t)| = \sqrt{\frac{81}{10}\sin^2 3t + \frac{81}{10}\cos^2 3t} = \frac{9}{\sqrt{10}}$

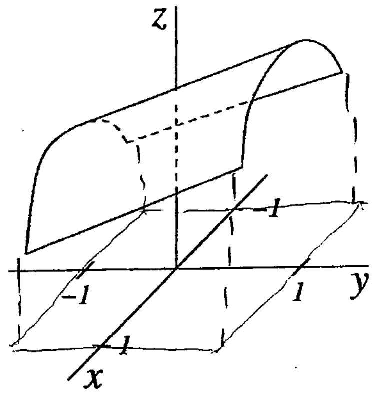
$\kappa(t) = \frac{|T'(t)|}{|r'(t)|} = \frac{\frac{9}{\sqrt{10}}}{\sqrt{10}} = \frac{9}{10}$

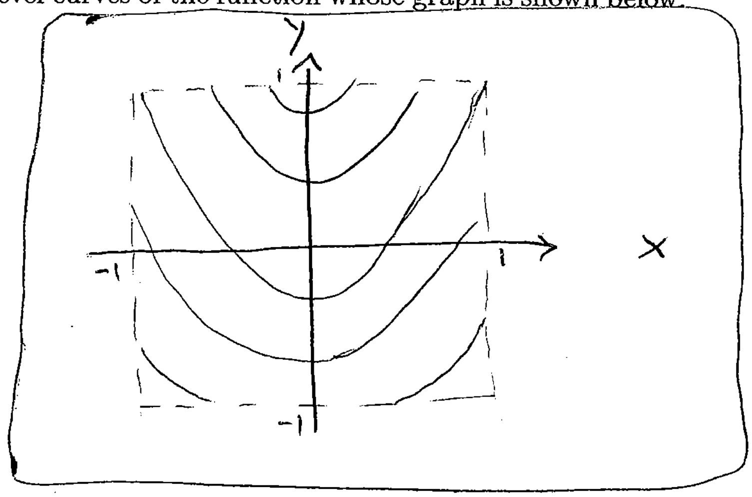
$$\boxed{\kappa(\pi) = \frac{9}{10}}$$

c) $N(t) = \frac{T'(t)}{|T'(t)|} = \frac{\sqrt{10}}{9}\left\langle \frac{-9}{\sqrt{10}}\sin 3t, 0, \frac{-9}{\sqrt{10}}\cos 3t \right\rangle$

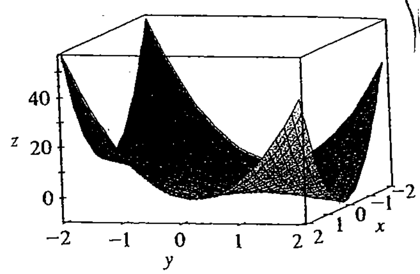
$$= \boxed{\langle -\sin 3t, 0, -\cos 3t \rangle}$$

2a. In an xy-coordinate system, sketch the level curves of the function whose graph is shown below.

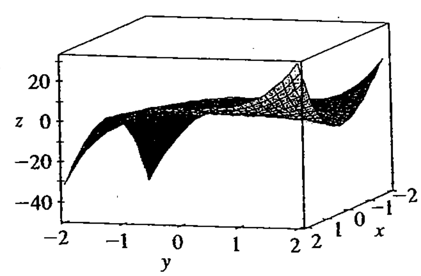




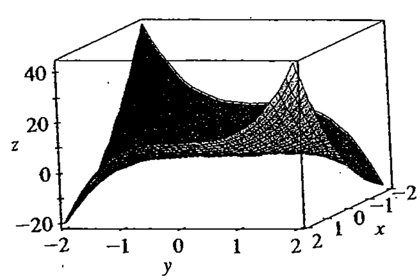
2b. The following surfaces are graphs of a function f and its partial derivatives f_x and f_y. Determine which is which and label the graphs accordingly, with brief explanations.



f_y



f



f_x

Try looking at curves on the front and sides of the boxes.

For the front, the f_y curve will be the derivative of the f curve.

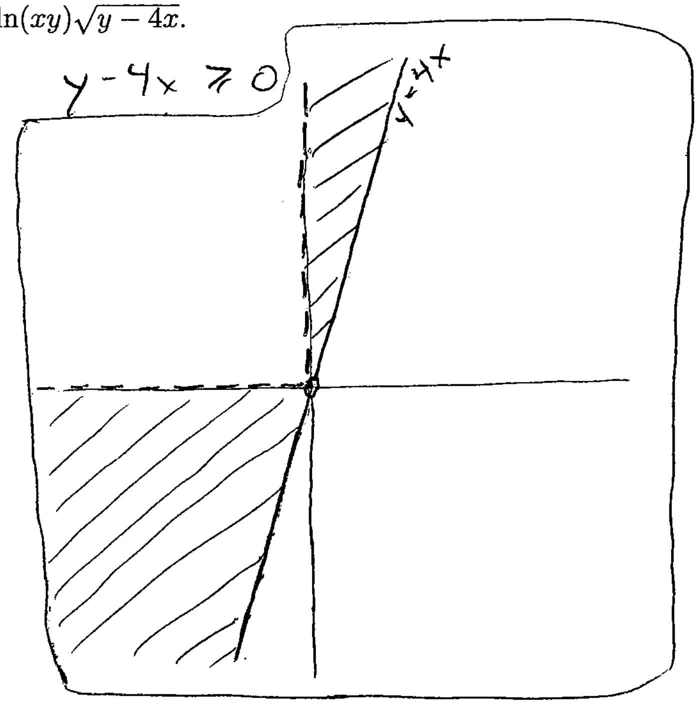
For the side, the f_x curve will be the derivative of the f curve.

3a. Sketch carefully the domain of $f(x,y) = x\ln(xy)\sqrt{y-4x}$.

Domain: $\underline{xy > 0}$ and $y - 4x \geq 0$

$\Updownarrow$

$x > 0$ and $y > 0$
OR $x < 0$ and $y < 0$



$\underline{xy > 0:}$ $\underline{y - 4x \geq 0:}$

3b. Find the linearization of $f(x,y) = x\sqrt{y}$ at the point $(1,4)$. Then use it to estimate $f(0.99, 4.003)$.

$\underline{f(1,4) = 2}$, $f_x(x,y) = \sqrt{y} \Rightarrow \underline{f_x(1,4) = 2}$

$f_y(x,y) = \dfrac{x}{2\sqrt{y}} \Rightarrow \underline{f_y(1,4) = \frac{1}{4}}$.

So $\boxed{L(x,y) = 2 + 2(x-1) + \frac{1}{4}(y-4)}$.

Then $f(0.99, 4.003) \approx L(0.99, 4.003)$

$= 2 + 2(-.01) + \frac{1}{4}(.003)$

$= 2 - .02 + .00075$

$= \boxed{1.98075}$

(actual value $\approx 1.980742361\cdots$)

4a. Find an equation of the tangent plane to the surface $x = y^2 + z^2 - 2$ at the point $(-1, 1, 0)$.

Let $F(x,y,z) = -x + y^2 + z^2$, then surface is $F(x,y,z) = 2$.

$\nabla F = \langle -1, 2y, 2z \rangle$, so normal vector is $\nabla F(-1,1,0)$

$= \underline{\langle -1, 2, 0 \rangle}$.

Tangent Plane:

$$\boxed{-(x+1) + 2(y-1) = 0}$$

4b. Five numbers x, y, z, u, and v are multiplied together. The first two are increasing at 5 units per second, and the last three are decreasing at 3 units per second. Find the rate of change of the product at a moment when each of the numbers is 10.

$f(x,y,z,u,v) = xyzuv$.

$$\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial x}}_{yzuv}\underbrace{\frac{dx}{dt}}_{5} + \underbrace{\frac{\partial f}{\partial y}}_{xzuv}\underbrace{\frac{dy}{dt}}_{5} + \underbrace{\frac{\partial f}{\partial z}}_{xyuv}\underbrace{\frac{dz}{dt}}_{-3} + \underbrace{\frac{\partial f}{\partial u}}_{xyzv}\underbrace{\frac{du}{dt}}_{-3} + \underbrace{\frac{\partial f}{\partial v}}_{xyzu}\underbrace{\frac{dv}{dt}}_{-3}$$

when all numbers $= 10$, we have

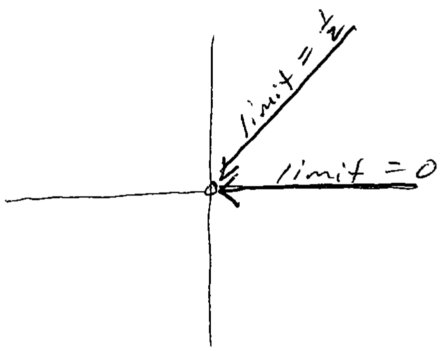
$$\frac{df}{dt} = 10^4 \cdot 5 + 10^4 \cdot 5 + 10^4(-3) + 10^4(-3) + 10^4(-3)$$

$$= \boxed{10000}.$$

5a. Show that the limit $\underbrace{\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^3+3y^3}}_{f(x,y)}$ does not exist.

Try $y=0$: $f(x,0) = \frac{0}{x}$, zero on x-axis

Try $y=x$: $f(x,x) = \frac{2x^3}{4x^3} = \frac{1}{2}$ on line $y=x$.



Since two paths to $(0,0)$ have different limiting values, the limit does not exist.

5b. Consider the functions $f(x,y) = \frac{x^2+2}{1+2y^2}$ and $g(x,y) = \frac{x^3+2x}{x+2y^2x}$. $\simeq \frac{x(x^2+2)}{x(1+2y^2)}$

(i) Where is f continuous? Where is g continuous?

(ii) Does $\lim_{(x,y)\to(0,0)} g(x,y)$ exist? Explain carefully why or why not. [Hint: compare with f.]

i) f is continuous everywhere, since denominator is never zero.

g is continuous when $x \neq 0$, i.e. away from the y-axis.

ii) f and g agree on the domain of g, so

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} g(x,y).$$

The former limit is $f(0,0)$ since f is continuous.

So the limit exists, and equals $f(0,0) = 2$.

6a. Find the directional derivative of $\frac{1}{xz} + \frac{1}{yz}$ at $(2, 2, 1)$ in the direction of the origin.

$$\underline{u} = \frac{\langle -2,-2,-1\rangle}{|\langle -2,-2,-1\rangle|} = \underline{\left\langle \frac{-2}{3}, \frac{-2}{3}, \frac{-1}{3} \right\rangle}.$$

Let $F(x,y,z) = \frac{1}{xz} + \frac{1}{yz}$, then $\nabla F = \left\langle \frac{-1}{x^2 z}, \frac{-1}{y^2 z}, \frac{-1}{xz^2} + \frac{-1}{yz^2} \right\rangle$.

So $\nabla F(2,2,1) = \left\langle \frac{-1}{4}, \frac{-1}{4}, -1 \right\rangle$.

$$D_{\underline{u}} F(2,2,1) = \nabla F \cdot \underline{u} = \left(\tfrac{-1}{4}\right)\left(\tfrac{-2}{3}\right) + \left(\tfrac{-1}{4}\right)\left(\tfrac{-2}{3}\right) + (-1)\left(\tfrac{-1}{3}\right)$$
$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \boxed{\frac{2}{3}}.$$

6b. Find the maximum rate of change of $x^2y^3z^4$ at $(x, y, z) = (1, 1, 1)$, and the direction in which it occurs.

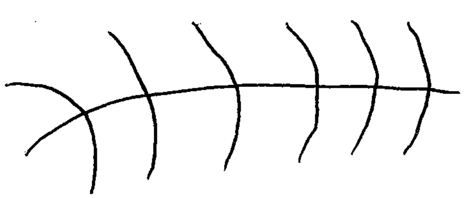
$F(x,y,z) = x^2y^3z^4$, $\nabla F = \langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle$

So $\nabla F(1,1,1) = \langle 2, 3, 4 \rangle$.

Max. rate of change $= |\nabla F| = \sqrt{4+9+16} = \boxed{\sqrt{29}}$.

Direction $=$ dir. of ∇F $= \frac{\langle 2,3,4\rangle}{|\langle 2,3,4\rangle|} = \boxed{\left\langle \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \right\rangle}$.

Bonus (+2 points) Here is a map showing a mountain stream and some contour lines. Which way is the stream flowing, and why?



The question is, which side is higher?

Left higher $\Rightarrow$ stream runs along the top of a ridge (not likely)

Right higher $\Rightarrow$ stream runs along the bottom of a valley (likely)

So stream most likely flows to the left.