Name: Id:

Final Exam
Math 2443-006
May 7, 2007

Do seven problems

Problem 1: Problem 5:

Problem 2: Problem 6:

Problem 3: Problem 7:

Problem 4: Problem 8:

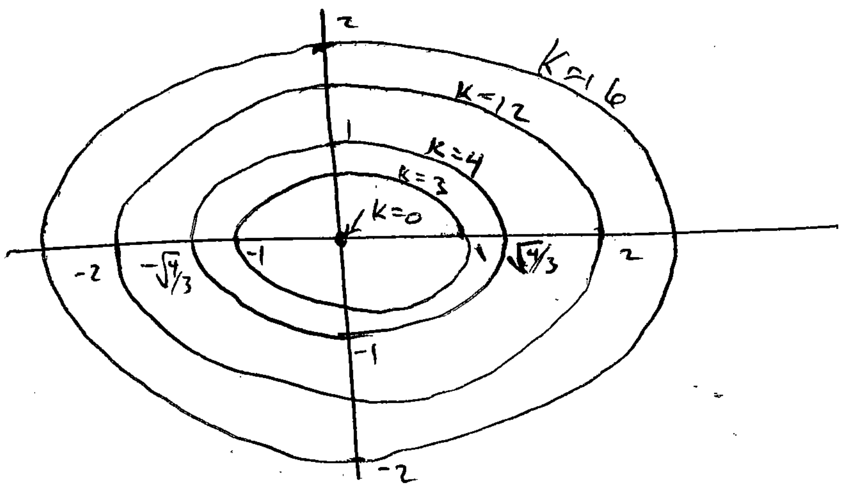
Total:

1. **(a)** Draw carefully some of the level curves for $f(x,y) = 3x^2 + 4y^2$, and indicate the levels.

curves: $3x^2 + 4y^2 = k$ for various k

or, $\frac{3}{k}x^2 + \frac{4}{k}y^2 = 1 \Rightarrow$ ellipses, with intercepts

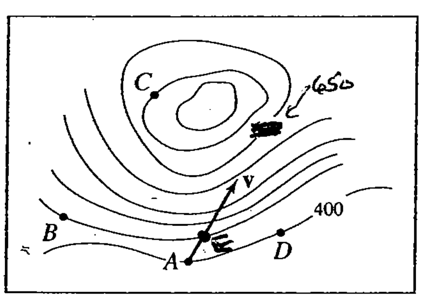
$(\pm\sqrt{k/3}, 0)$, $(0, \pm\sqrt{k/4})$



(b) For the contour map below, estimate the following quantites. Consider the units carefully.

(i) The average rate of change of elevation from A to C and from A to D.

(ii) The directional derivative at A in the direction **v**.



Contour interval = 50 meters

0 1 2 km

(i) want $\dfrac{\text{change in elev.}}{\text{change in pos'n.}}$

A to C: $\dfrac{300\text{ m}}{2000\text{ m}} = \boxed{0.15}$ avg. rate of change

A to D: $\dfrac{0\text{ m}}{1000\text{ m}} = \boxed{0}$

(ii) $D_{\underline{v}}f = \nabla f \cdot \underline{v}$

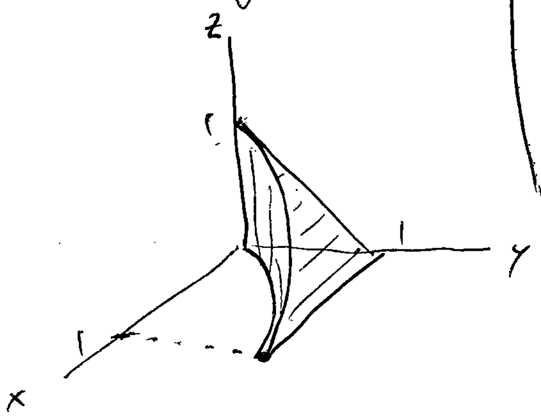
also = rate of change in direction $\underline{v}$. Let's use rate of change from A to E.

$= \dfrac{\text{change in elev.}}{\text{distance}} = \dfrac{50\text{ m}}{400\text{ m}} = \boxed{0.125}$

2. Set up (but *do not evaluate*) integrals over the following regions.

(a) $\iiint_E x^2y^2\,dV$ where E is bounded by $y=\sqrt{x}$, $z=1-y$, $z=0$, and $x=0$.

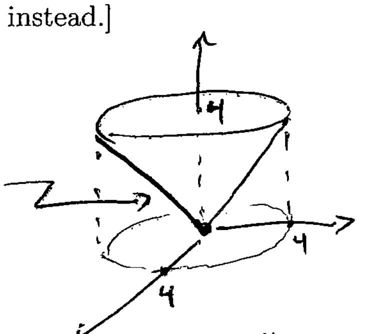
one way is:

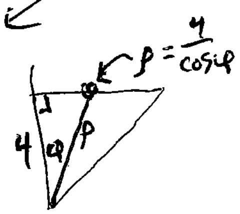


$$\boxed{\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} x^2y^2\,dz\,dy\,dx}$$

(b) $\iiint_E (x+yz)\,dV$ where E is the region above the cone $z=\sqrt{x^2+y^2}$ and below the plane $z=4$, in cylindrical coordinates. [**Extra credit:** +2 if you do it with spherical coordinates instead.]

cone: $z=r$





cylindrical:

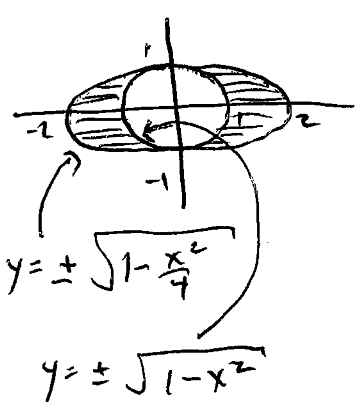
$$\int_0^{2\pi}\int_0^4\int_r^4 (r\cos\theta + rz\sin\theta)\,r\,dz\,dr\,d\theta$$

spherical:

$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{4\sec\varphi} (\rho\sin\varphi\cos\theta + \rho^2\sin\varphi\sin\theta\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

(b) $\iint_D y\cos x\,dA$ where D is the region between the unit circle and the ellipse $\frac{x^2}{4}+y^2=1$.

one way is: ellipse − circle



$$\int_{-2}^{2}\int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} y\cos x\,dy\,dx - \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\cos x\,dy\,dx$$

3. **(a)** Give two examples of vector fields $\langle P, Q\rangle$ such that $Q_x - P_y = 1$. Then give the corresponding formulas for area that follow by Green's Theorem.

$$\boxed{\langle P, Q\rangle = \langle 0, x\rangle \implies \text{Area} = \iint_D dA \underset{G.T.}{=} \int_C 0\,dx + x\,dy}$$

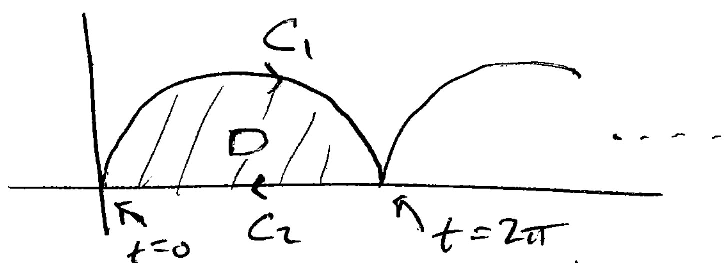
$Q_x - P_y = 1 - 0$ ✓

$$\boxed{\langle P, Q\rangle = \langle -y, 0\rangle \implies \text{Area} = \iint_D dA \underset{G.T.}{=} \int_C -y\,dx + 0\,dy}$$

$Q_x - P_y = 0 - (-1)$ ✓

(b) Use one of the formulas above to find the area under one arch of the cycloid parametrized by $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$.

cycloid:



$C = C_1$ and C_2

Note C is oriented backwards, so area formula must be modified by a $-$ sign.

$$\text{So Area} = \int_{C_1} y\,dx + \int_{C_2} y\,dx = \int_0^{2\pi} \underbrace{(1-\cos t)}_{y}\underbrace{(1-\cos t)\,dt}_{dx} + \int_{C_2} 0 \cdot dx$$

$y = 0$ on C_2

$$= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt = \int_0^{2\pi} \left(1 - 2\cos t + \tfrac{1}{2}(1 + \cos 2t)\right) dt$$

$$= \int_0^{2\pi} \left(\tfrac{3}{2} - 2\cos t + \tfrac{1}{2}\cos 2t\right) dt$$

zero from 0 to 2π

$$= \tfrac{3}{2} \cdot 2\pi = \boxed{3\pi}$$

4. **(a)** Use implicit differentiation to find $\frac{\partial w}{\partial z}$ at the point $(x, y, z) = (2, 1, 4)$, where $\frac{1}{w} = \frac{1}{xy} + \frac{1}{yz}$.

$$\frac{\partial}{\partial z}\left(\frac{1}{w}\right) = \frac{\partial}{\partial z}\left(\frac{1}{xy} + \frac{1}{yz}\right)$$

$$\frac{1}{w} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\frac{-1}{w^2}\frac{\partial w}{\partial z} = 0 + \frac{-1}{yz^2}$$

$$\frac{\partial w}{\partial z} = \frac{w^2}{yz^2}$$

at $(2,1,4)$, $w = \frac{4}{3}$ so $\frac{\partial w}{\partial z} = \frac{16/9}{16} = \boxed{\frac{1}{9}}$

(b) The length ℓ, the width w, and the height h (in meters) of a box change with time. At a certain instant the dimensions are $\ell = 4$, $w = 3$, $h = 10$, and ℓ and w are increasing at a rate of 3 m/s, and h is decreasing at a rate of 5 m/s. At what rate is the surface area changing?

Area $= 2(\ell w + wh + \ell h)$

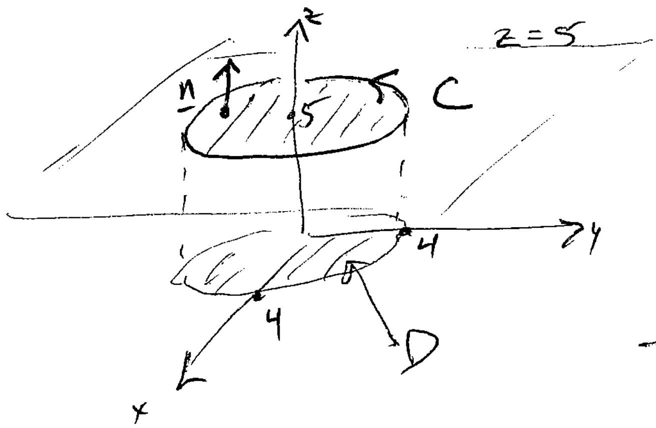
$$\frac{dA}{dt} = \frac{\partial A}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial A}{\partial w}\frac{dw}{dt} + \frac{\partial A}{\partial h}\frac{dh}{dt}$$

$$= (\underset{\substack{\uparrow\\3}}{2w} + \underset{\substack{\uparrow\\10}}{2h})\underset{\substack{\uparrow\\3}}{\frac{d\ell}{dt}} + (\underset{\substack{\uparrow\\4}}{2\ell} + \underset{\substack{\uparrow\\10}}{2h})\underset{\substack{\uparrow\\3}}{\frac{dw}{dt}} + (\underset{\substack{\uparrow\\3}}{2w} + \underset{\substack{\uparrow\\4}}{2\ell})\underset{\substack{\uparrow\\-5}}{\frac{dh}{dt}}$$

$$= 26 \cdot 3 + 28 \cdot 3 + 14(-5)$$

$$= \boxed{92 \text{ m}^2/\text{sec}.}$$

5. Use Stokes' Theorem to evaluate $\int_C (yz\mathbf{i} + 2xz\mathbf{j} + e^{xy}\mathbf{k}) \cdot d\mathbf{r}$ where C is the circle $x^2 + y^2 = 16$, $z = 5$, oriented counterclockwise as viewed from above.



Let S = part of plane $z=5$ inside C.

Then $\underline{n} = \langle 0, 0, 1 \rangle$.

Parametize with x, y:

$\underline{r}(x,y) = \langle x, y, 5 \rangle$

$\underline{r}_x = \langle 1, 0, 0 \rangle$

$\underline{r}_y = \langle 0, 1, 0 \rangle$

$\underline{r}_x \times \underline{r}_y = \langle 0, 0, 1 \rangle$

so $dS = |\underline{r}_x \times \underline{r}_y|\, dA = dx\,dy$.

Now:

$$\int_C \langle yz, 2xz, e^{xy} \rangle \cdot d\underline{r} \underset{\text{Stokes}}{=} \iint_S \operatorname{curl} \langle yz, 2xz, e^{xy} \rangle \cdot \underline{n}\, dS$$

$$= \iint_S \langle \text{—}, \text{—}, 2z - z \rangle \cdot \langle 0, 0, 1 \rangle\, dS$$

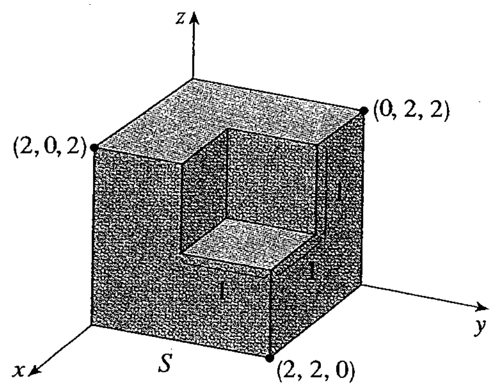
$$= \iint_S z\, dS = \iint_S 5\, dS$$

$$= \iint_D 5\, dx\,dy$$

$$= 5 \cdot \text{Area}(D)$$

$$= 5 \cdot 16\pi = \boxed{80\pi}$$

6. **(a)** Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the outwardly oriented surface shown below (the boundary surface of a cube with a unit corner cube removed).



$$\iint_S \underline{F} \cdot \underline{n}\, dS = \iiint_E \operatorname{div} \underline{F}\, dV$$

where E = solid región.

$$\operatorname{div} \underline{F} = 1 + 1 + 1 = 3$$

So integral $= \iiint_E 3\, dV$

$$= 3 \cdot \text{Vol}(E)$$

$$= 3 \cdot 7$$

$$= \boxed{21}$$

(b) Is $\mathbf{G}(x, y, z) = \langle e^{x-y}, e^{y-z}, e^{z-x} \rangle$ equal to $\operatorname{curl} \mathbf{F}$ for some vector field $\mathbf{F}$? Explain why or why not, and find $\mathbf{F}$ if it exists.

$\operatorname{div}(\operatorname{curl} \underline{F}) = 0$, always.

Now $\operatorname{div} \underline{G} = e^{x-y} + e^{y-z} + e^{z-x} \neq 0$

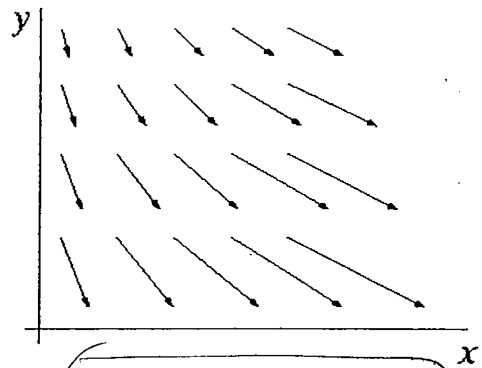
So $\underline{G}$ can't be curl $\underline{F}$ for any $\underline{F}$.

7. **(a)** The vector field $\langle P, Q\rangle$ in the xy-plane is shown below. Let $\mathbf{F}(x,y,z) = \langle P, Q, 0\rangle$.

(i) Determine which of P_x, P_y, Q_x, and Q_y are positive, negative, or zero.

(ii) Say what you can about $\operatorname{curl}\mathbf{F}$.

(iii) Say what you can about $\operatorname{div}\mathbf{F}$.



(i) $\boxed{P_x > 0, \quad P_y < 0 \qquad Q_x = 0, \quad Q_y > 0}$

(note – Q is negative in the picture)

(ii) $\operatorname{curl} \underline{F} = \langle 0, 0, Q_x - P_y \rangle = \boxed{\langle 0, 0, \text{positive} \rangle}$

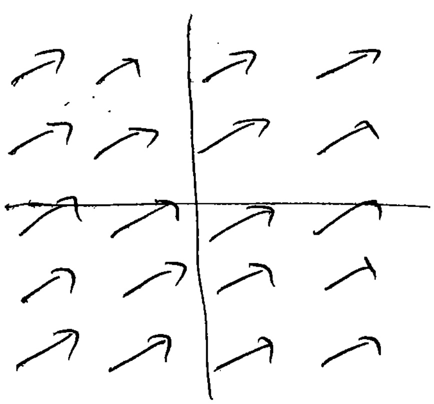
(also, counterclockwise rotation is evident in picture)

(iii) $\operatorname{div} \underline{F} = P_x + Q_y \ \boxed{> 0}$

(at any point, more leaving than entering)

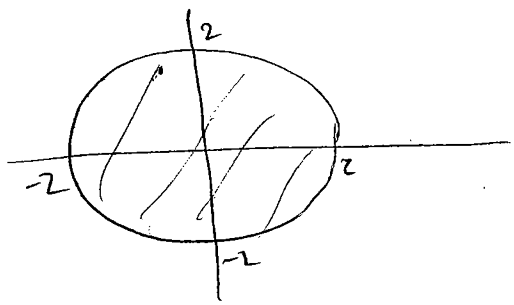
(b) Draw carefully a vector field $\mathbf{F}$ in the plane, with the property that for any curve C, the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the endpoints of C. Explain why $\mathbf{F}$ has this property.

<u>many possibilities</u> – any ∇f will work, or any $\langle P, Q\rangle$ with $Q_x = P_y$ (such as a constant vec. field). These are conservative, and



$\int_C \underline{F}\cdot d\underline{r}$ depends only on endpoints for such $\underline{F}$.

8. Find the location(s) and the value of the absolute minimum of $f(x,y) = x^2 - y^2$ on the disk $x^2 + y^2 \leq 4$.



Interior

$$\left.\begin{aligned} f_x &= 2x = 0 \\ f_y &= -2y = 0 \end{aligned}\right\} \Rightarrow (x,y) = (0,0)$$

the only critical point.

boundary: parametrize as $\underline{r}(t) = \langle 2\cos t, 2\sin t \rangle$.

Want extreme values of $\underbrace{f(2\cos t, 2\sin t)}_{g(t)} = 4\cos^2 t - 4\sin^2 t$.

$g'(t) = -8\cos t \sin t - 8 \sin t \cos t = -16 \sin t \cos t$

zero when $\sin(t) = 0$ or $\cos(t) = 0$

i.e. $t = 0, \pi/2, \pi, 3\pi/2$, etc.

So boundary points to check are these, i.e.

$(2,0), (0,2), (-2,0), (0,-2)$.

final check:

$f(0,0) = 0$
$f(2,0) = 4$
$f(0,2) = -4$
$f(-2,0) = 4$
$f(0,-2) = -4$

$$\boxed{\begin{array}{l} \text{min value} = -4 \\ \text{at } (0, \pm 2) \end{array}}$$