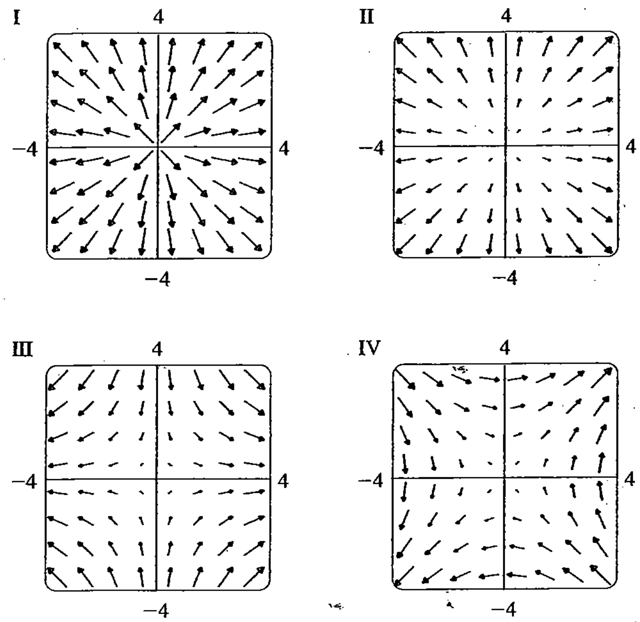
1. (8 points) Match the vector fields **F** with the pictures I–IV:

IV (a) $\mathbf{F}(x,y) = \langle y, x \rangle$

III (b) $\mathbf{F}(x,y) = \langle 2x, -2y \rangle$

II (c) $\mathbf{F}(x,y) = \langle 2x, 2y \rangle$

I (d) $\mathbf{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$



2. (8 points) Are the following vector fields **F** conservative? If so find a function f with $\nabla f = \mathbf{F}$, and otherwise give a reason why not.

(a) $\mathbf{F}(x,y) = \langle 3x \sin y, x^2 \cos y \rangle$

$P \quad Q$

$\frac{\partial P}{\partial y} = 3x\cos y \quad , \quad \frac{\partial Q}{\partial x} = 2x \cos y$

$\boxed{\text{not equal, so } F \text{ is not conservative}}$

(b) $\mathbf{F}(x,y) = \langle 2x \cos y, -x^2 \sin y - 4 \rangle$

$P \quad Q$

$\frac{\partial P}{\partial y} = -2x\sin y \quad , \quad \frac{\partial Q}{\partial x} = -2x \sin y$ ✓

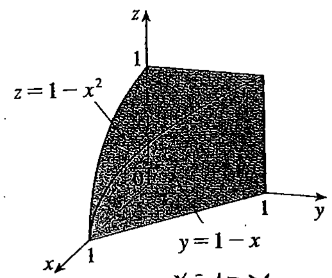
Also F is defined on $\mathbb{R}^2$. $\boxed{F \text{ is conservative}}$

$f_x = 2x\cos y \Rightarrow f = x^2 \cos y + C(y)$

$f_y = -x^2 \sin y - 4 \Rightarrow f = x^2\cos y - 4y + D(x)$

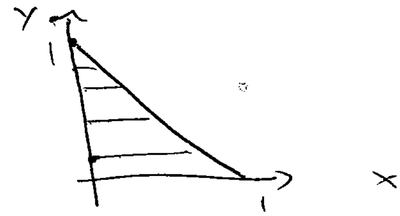
So $\boxed{f(x,y) = x^2 \cos y - 4y + E}$

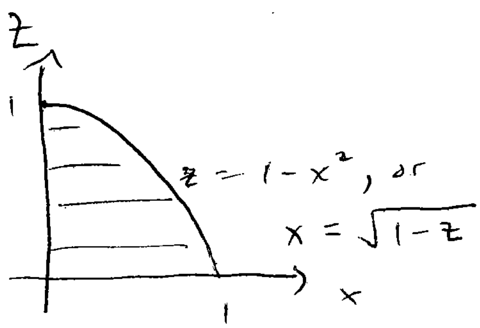
3. (8 points) The region E is bounded by the surface $z = 1 - x^2$, the plane $y = 1 - x$, and the three coordinate planes. Express the integral $\iiint_E f(x, y, z)\, dV$ in three different ways, using $dV = dz\, dx\, dy$, $dV = dy\, dx\, dz$, and $dV = dy\, dz\, dx$.



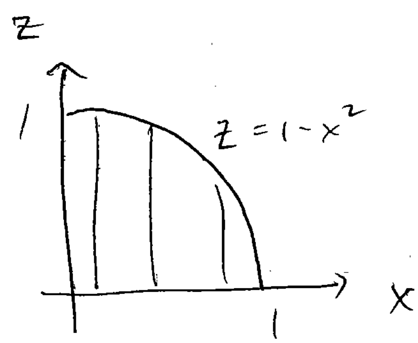
$x = 1 - y$

$$\boxed{\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\, dz\, dx\, dy}$$





$$\boxed{\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\, dy\, dx\, dz}$$



$$\boxed{\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\, dy\, dz\, dx}$$

4. (8 points) Evaluate the following line integrals:

(a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle y^2, xy \rangle$ and C is given by $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $0 \le t \le 4$.

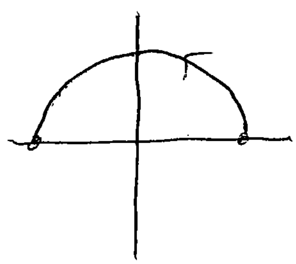
$$\int_0^4 \langle (t^3)^2, (t^2)(t^3) \rangle \cdot \langle 2t, 3t^2 \rangle \, dt$$

$$= \int_0^4 2t^7 + 3t^7 \, dt = \int_0^4 5t^7 \, dt$$

$$= \frac{5}{8} t^8 \Big|_0^4$$

$$= \boxed{\frac{5}{8} 4^8}$$

(b) $\int_C xy^2 \, ds$ where C is the upper half of the unit circle $x^2 + y^2 = 1$.



$r(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le \pi$

$r'(t) = \langle -\sin t, \cos t \rangle$

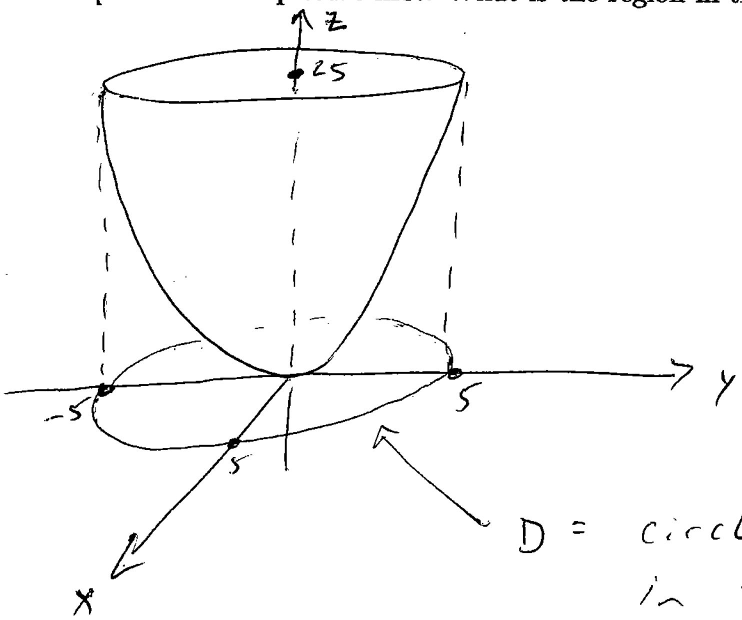
$ds = \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt = dt$

$$\int_C xy^2 \, ds = \int_0^\pi \cos t \sin^2 t \, dt$$

$u = \sin t, \quad du = \cos t \, dt$

$$\int_0^0 u^2 \, du = \boxed{0}.$$

5. (8 points) Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 25$. [Hint: draw a picture first. What is the region in the xy–plane?]



D = circle of radius 5 in x-y plane

$$z = f(x,y) = x^2 + y^2$$

$$f_x = 2x$$

$$f_y = 2y$$

$$SA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

polar coords

$$= \int_0^{2\pi}\int_0^5 \sqrt{1 + 4r^2\cos^2\theta + 4r^2\sin^2\theta}\; r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^5 \sqrt{1+4r^2}\; r\,dr\,d\theta$$

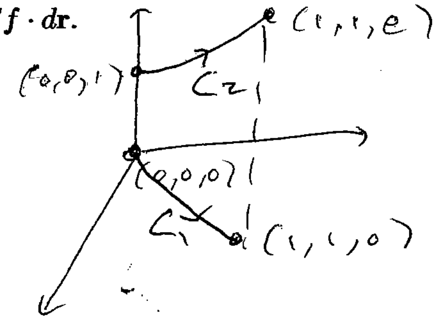
$u = 1 + 4r^2$
$du = 8r\,dr$

$$= \int_0^{2\pi}\int_1^{101} \tfrac{1}{8} u^{1/2}\,du\,d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{12} u^{3/2}\Big|_1^{101}\, d\theta$$

$$= 2\pi\left(\tfrac{1}{12}\left(101^{3/2} - 1\right)\right) = \boxed{\frac{\pi}{6}\left(101^{3/2} - 1\right)}.$$

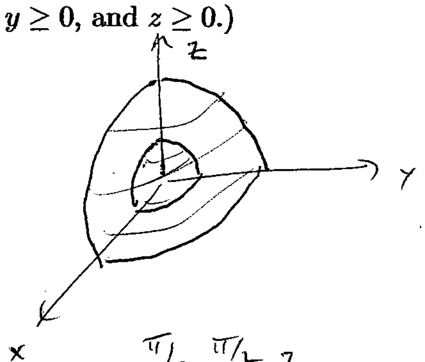
6. (6 points) Let $f(x,y,z)=\cos(x^2+2y-z)$. Let C_1 be the line segment from $(0,0,0)$ to $(1,1,0)$ and let C_2 be the curve on the surface $z=e^{xy}$ lying directly above C_1. Find $\int_{C_1}\nabla f\cdot d\mathbf{r}$ and $\int_{C_2}\nabla f\cdot d\mathbf{r}$.



$$\int_{C_1}\nabla f\cdot d\underline{r} = f(1,1,0)-f(0,0,0)$$
$$=\boxed{\cos(3)-1}.$$

$$\int_{C_2}\nabla f\cdot d\underline{r} = f(1,1,e)-f(0,0,1)$$
$$=\boxed{\cos(3-e)-\cos(-1)}$$

7. (8 points) Evaluate $\iiint_E z\,dV$ where E is the region between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$ in the first octant. (Recall that the *first octant* means the region where $x\geq 0$, $y\geq 0$, and $z\geq 0$.)



spherical coords:

$$\left.\begin{array}{l} 1\leq\rho\leq 2\\ 0\leq\theta\leq \pi/2\\ 0\leq\varphi\leq\pi/2\end{array}\right\} E$$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (z)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^3\cos\varphi\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\cos\varphi\sin\varphi\,d\varphi\int_1^2\rho^3\,d\rho$$

$u=\sin\varphi$
$du=\cos\varphi\,d\varphi$

$$=\frac{\pi}{2}\int_0^1 u\,du\left(\frac{1}{4}\rho^4\Big|_1^2\right)$$

$$=\frac{\pi}{2}\left(\frac{1}{2}u^2\Big|_0^1\right)\left(\frac{1}{4}2^4-\frac{1}{4}\right) = \frac{\pi}{16}(2^4-1)=\boxed{\frac{15\pi}{16}}$$